Supplementary Material for
“Patent Auctions and Bidding Coalitions: Structuring the Sale of Club Goods”

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Abstract

We present results pertaining to the complete information case of the model studied in the paper, and to an alternative stability notion that allows for joint deviations.

1 The Complete Information Case

This section provides a characterization of the stable coalition profiles in the complete information case, in which every firm's value for the patent is known and equal across all firms. In this situation, we show that any coalition profile in which there are at least two coalitions strictly larger than $\overline{n}$ is stable, and allows the seller to extract the maximum surplus. However, there are other stable coalition profiles (for example, the grand coalition), in which the seller's revenue can be lower, and potentially zero. Notably, we show that the maximum revenue that the seller can extract from the auction is non-monotonic in the number of applications $\overline{n}$–that is, sellers benefit from developing technologies with an intermediate number of applications.

To start the analysis, we assume that all firms' valuations are known and equal to $v > 0$. This setup also applies when firms' private valuations are realized after the auction takes place and v is the individual expected patent's value. Note that, with complete information, the limited-values case and the optimized-values one are clearly equivalent, and the total value of the patent for a coalition σ_j of size n_j is always $W_j = \min\{n_j, \overline{n}\} \cdot v$.

The following lemma describes the equilibrium payoff arising in the auction stage for any given coalition profile:

Lemma A1 (Coalitions' Payoffs with Complete Information) *For any coalition profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_J)$, the payoff of each coalition σ_j is*

$$\pi(\sigma_j;\sigma) = \begin{cases} (\min\{n_1, \overline{n}\} - \min\{n_2, \overline{n}\})v & \text{if } j = 1, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Lemma A1 follows immediately from the observation that, in the complete information case, the auction yields a strictly positive payoff to the winner only if $n_2 < \min\{n_1, \overline{n}\}$. In fact, if either $n_1 = n_2 < \overline{n}$ or $n_2 \geq \overline{n}$, the patent's valuation for the two largest coalitions is the same, yielding zero payoff for the winner.

The next result follows from Lemma A1 and characterizes the stable coalition profiles.[1]

Proposition A1 (Stable Coalition Profiles with Complete Information) *With complete information, a coalition profile σ is stable if and only if it satisfies either (i) $n_1, n_2 > \overline{n}$, or (ii) $n_1 \geq \overline{n} \geq n_2 = n_3$.*

Proof of Proposition A1: Consider a coalition profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_J)$ and let $n_j \equiv 0$ for $j > J$. Consider the following cases:

(1) If $n_1 < \overline{n}$, a firm $i \in \sigma_2$ has a profitable deviation in joining σ_1. As $\overline{n} \leq N$ and $\overline{n} > n_1 \geq n_2 > 0$,

$$\pi(\sigma_1;\sigma) + \pi(\sigma_2;\sigma) = (n_1 - n_2)v < [(n_1 + 1) - (n_2 - 1)]v = \pi(\sigma_1';\sigma') + \pi(\sigma_2';\sigma')$$

with $\sigma_1' = \sigma_1 \cup \{i\}$, $\sigma_2' = \sigma_2 \backslash \{i\}$, and $\sigma_k' = \sigma_k$ for $k \neq 1, 2$.

(2) If $n_1 \geq n_2 > \overline{n}$, any coalition profile is stable. No unilateral deviation by any firm to any coalition changes the payoff of the winning coalition and the losing coalitions as they all remain equal to zero.

(3) If $n_1 \geq \overline{n} \geq n_2 > n_3$, a firm i in σ_2 has a profitable deviation in joining σ_1 because

$$\pi(\sigma_1;\sigma) + \pi(\sigma_2;\sigma) = (\overline{n} - n_2)v < [\overline{n} - (n_2 - 1)]v = \pi(\sigma_1';\sigma') + \pi(\sigma_2';\sigma'),$$

with $\sigma_1' = \sigma_1 \cup \{i\}$, $\sigma_2' = \sigma_2 \backslash \{i\}$, and $\sigma_k' = \sigma_k$ for $k \neq 1, 2$.

(4) If $n_1 \geq \overline{n} \geq n_2 = n_3$, no firm $i \in \sigma_j$ has a profitable deviation in deviating to be a singleton. Since the coalition $\{i\}$ never wins the patent auction with a strictly positive profit,

[1] The proof of Proposition A1 is presented in Section 6.3.

we have

$$\pi(\sigma_j;\sigma) = \begin{cases} (\overline{n} - n_2)v & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

cannot be strictly lower than

$$\pi(\sigma'_j;\sigma') + \pi(\{i\};\sigma') = \pi(\sigma'_j;\sigma') = \begin{cases} (\min\{n_1 - 1, \overline{n}\} - n_2)v & \text{if } j = 1 \text{ and } n_1 > n_2 \\ 0 & \text{otherwise.} \end{cases}$$

with $\sigma'_j = \sigma_j \backslash \{i\}$, $\sigma'_{J+1} = \{i\}$, and $\sigma'_k = \sigma_k$ for $k \neq j$.

Next, consider a unilateral deviation by a firm $i \in \sigma_j$ to join another coalition σ_k, to form $\sigma'_j = \sigma_j \backslash \{i\}$, $\sigma'_k = \sigma_k \cup \{i\}$, and $\sigma'_h = \sigma_h$ for $h \neq j, k$. If $j, k \neq 1$, the deviation would not be profitable, as since $n_1 \geq \overline{n}$, neither σ_j or σ_k, and neither σ'_j or σ'_k, win the patent auction with a strictly positive payoff. On the other hand, if $j = 1$ or $k = 1$, then the winning coalition's payment to the seller can only be (weakly) increased by the deviation due to the fact that $n_2 = n_3$. As the winning coalition's valuation is bounded above by $\overline{n}v$, we have

$$\pi(\sigma_j;\sigma) + \pi(\sigma_k;\sigma) = \pi(\sigma_1;\sigma) = \overline{n}v - n_2 v \geq \pi(\sigma'_j;\sigma') + \pi(\sigma'_k;\sigma'). \quad \blacksquare$$

Proposition A1 guarantees the existence of a stable coalition profile with complete information: the grand coalition (i.e., $\sigma = \{N\}$) is always stable as it satisfies condition (ii). On the other hand, a stable coalition profile identified by condition (i) exists if and only if $\overline{n} \leq \frac{N}{2} - 1$. Thus, when $\overline{n}$ is large relative to N (i.e., when the new technology has a large number of applications), the set of stable coalition profiles tends to be smaller and may contain only the grand coalition.

Two additional observations follow from Proposition A1. First, if all firms bid individually without the option of forming coalitions, we have $\sigma_j = \{j\}$ and $\pi(\sigma_j;\sigma) = 0$ for any $j = 1, .., N$, and the seller's revenue would be v. Thus, as long as $n_2 > 1$, the seller can be better off in the presence of coalitions than in their absence. Such an increase in the seller's revenue is due to the club-good nature of the patent. When the patent has potential multiple applications ($\overline{n} > 1$), a coalition can generate more value from it than what an individual firm would. However, for the seller to be able to extract the surplus $\overline{n}v$ in full through an auction, at least two coalitions larger than $\overline{n}$ need to form and compete for the patent, so we must have $\overline{n} \leq \frac{N}{2} - 1$.

Second, the fact that a patent (i.e., a club good) is the object being sold at the auction generates an additional incentive for bidders to cooperate. Consider the largest coalition in any given coalition profile. Even if this coalition is already larger than $\overline{n}$, the coalition might have an incentive to enlarge its size by adding firms belonging to the second-largest coalition. In fact, as long as $n_2 \leq \overline{n}$, by reducing the size of the second-largest coalition, the largest coalition lower its value, which is the price paid by the largest coalition. For example, consider a coalition profile such that $n_1 \geq \overline{n} \geq n_2 > n_3$. Such profile is clearly not stable because a firm $i \in \sigma_2$ has a profitable deviation in joining σ_1 as

$$\pi(\sigma_1 \cup \{i\}; \sigma') + \pi(\sigma_2 \backslash \{i\}; \sigma') = \pi(\sigma_1 \cup \{i\}; \sigma') = \overline{n}v - (n_2 - 1)v > \overline{n}v - n_2 v = \pi(\sigma_1; \sigma) + \pi(\sigma_2; \sigma),$$

where σ' corresponds to $\sigma'_1 = \sigma_1 \cup \{i\}$, $\sigma'_2 = \sigma_2 \backslash \{i\}$, and $\sigma'_i = \sigma_i$ for any $i \neq 1, 2$.

The next result illustrates the implications of Proposition A1 on the seller's revenue. Recall that we defined $R^*(N, \overline{n})$ as the maximum revenue achievable by a stable coalition profile for any given N and $\overline{n}$.

Corollary A1 (Seller's Revenue with Complete Information) *With complete information, if $\overline{n} \leq \frac{N}{2} - 1$, $R^*(N, \overline{n}) = \overline{n}v$, and if $\overline{n} > \frac{N}{2} - 1$, $R^*(N, \overline{n}) = \left\lfloor \frac{N - \overline{n}}{2} \right\rfloor v (\leq \overline{n}v)$. Therefore, $R^*(N, \overline{n})$ is non-monotonic in $\overline{n}$.*

Corollary A1 follows from the observation that in a stable coalition profile characterized by condition (i) in Proposition A1, the seller's revenue is $\overline{n}v$; in a stable coalition profile described by condition (ii), the seller's revenue is $n_2 v$, which is (weakly) less than $\overline{n}v$, and can potentially be zero in the grand coalition case. Corollary 1 also considers the seller's equilibrium revenue as a function of $\overline{n}$, which represents the scope of the patent's applications. If $\overline{n} \leq \frac{N}{2} - 1$, all the coalition profiles described in Proposition A1 are stable. Thus, at least two coalitions larger than $\overline{n}$ can be sustained in a stable coalition profile, generating a seller's revenue of $\overline{n}v$, which is increasing in $\overline{n}$, and maximized at $\overline{n} = \left\lfloor \frac{N}{2} - 1 \right\rfloor$. If $\overline{n} > \frac{N}{2} - 1$, the only stable coalition profiles are the ones characterized by (ii) of Proposition A1. The maximum possible revenue for the seller is achieved when n_2 and n_3, which have to be of the same, are as large as possible. This happens when $n_2 = n_3 = \left\lfloor \frac{N - \overline{n}}{2} \right\rfloor (\leq \overline{n} \leq n_1)$, yielding a seller's revenue of $\left\lfloor \frac{N - \overline{n}}{2} \right\rfloor v$, which is decreasing in $\overline{n}$. This implies that the maximum revenue achievable for the seller $R^*(N, \overline{n})$ is non-monotonic in $\overline{n}$, and patents are potentially most profitable when they have an intermediate number of applications relative to the total market size N (specifically, when $\overline{n} = \left\lfloor \frac{N}{2} - 1 \right\rfloor$).

Finally, we compare the seller's revenue raised by a multi-license ($(\overline{n}+1)$-th price) auction to the one generated by a second-price auction of a patent in the complete information case. The following result is an immediate implication of Corollary A1.

Proposition A2 (Dominance of Multi-license Auctions with Complete Information) *In the complete information case, a multi-license auction dominates a single-patent auction from the seller's perspective.*

2 Joint Deviations

In general, allowing for joint deviations in our setting would yield the grand coalition as a unique stable partition, with some exceptions. To see this, suppose that we allow joint deviations in which any set of firms belonging to the same coalition can deviate together and either form a new coalition, or join another coalition. The corresponding notion of stability is defined as follows.

Definition A1 (Profitable Joint Deviations) *Consider a coalition profile $\sigma = \{\sigma_1, .., \sigma_J\}$. A set of firms $\widetilde{\sigma} \subset \sigma_j \in \sigma$, has a profitable deviation if at least one of the following is true:*

(i) The coalition profile $\sigma' = (\sigma'_1, \ldots, \sigma'_{J+1})$ with $\sigma'_j = \sigma_j \backslash \widetilde{\sigma}$, $\sigma'_{J+1} = \widetilde{\sigma}$, and $\sigma'_k = \sigma_k$ for $k \neq j, h$ is such that[2]

$$\pi(\sigma_j; \sigma) < \pi(\sigma'_j; \sigma') + \pi(\widetilde{\sigma}; \sigma');$$

(ii) For some $k \neq j$, the coalition profile $\sigma' = (\sigma'_1, \ldots, \sigma'_J)$ with $\sigma'_j = \sigma_j \backslash \widetilde{\sigma}$, $\sigma'_k = \sigma_k \cup \widetilde{\sigma}$, and $\sigma'_h = \sigma_h$ for $h \neq j, k$ is such that

$$\pi(\sigma_j; \sigma) + \pi(\sigma_k; \sigma) < \pi(\sigma'_j; \sigma') + \pi(\sigma'_k; \sigma').$$

Definition A2 (Stable Coalition Profiles) *A coalition profile σ is stable if no set of firms has a profitable deviation.*

[2] For ease of notation, we deviate from our usual coalitions' numbering (with indexes weakly decreasing in coalitions" sizes) and we just assign the newly-formed coalition $\widetilde{\sigma}$ the index $J + 1$.

In the optimized-value case, the following result holds.

Proposition A3 *If joint deviations are allowed in the optimized-value case, the grand coalition is uniquely stable.*

Proof of Proposition A3: Take any coalition profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_J)$ such that σ_2 is non-empty. Consider a coalition profile σ' such that $\sigma'_1 = \sigma_1 \cup \sigma_2$, and $\sigma'_j = \sigma_{j+1}$ for any $j = 2, .., J-1$. Define W_1, W_2, and W'_1 be the values for coalitions σ_1, σ_2, and σ'_1, computed as described in Section 2.2. of the paper. Also, define Z be the highest value among all other coalitions in σ (or, equivalently in σ')–that is $Z \equiv \max_{j \neq 1,2}\{W_j\}$. Then,

$$\pi(\sigma_1; \sigma) + \pi(\sigma_2; \sigma) = \mathbf{E}[\max\{W_1, W_2, Z\} - \max\{\min\{W_1, W_2\}, Z\}]$$
$$\pi(\sigma'_1; \sigma') = \mathbf{E}[\max\{W'_1, Z\} - \min\{W'_1, Z\}].$$

Observe that every realization of W_1, W_2, and W'_1 satisfies $w'_1 \geq \max\{w_1, w_2\}$. Assume first that $w_1 \geq w_2$. The realizations of the payoffs have to satisfy one of the following cases:

(i) Suppose that $z \geq w'_1 \geq w_1 \geq w_2$. Then,

$$\max\{w_1, w_2, z\} - \max\{\min\{w_1, w_2\}, z\}] = 0$$
$$\leq \max\{w'_1, z\} - \min\{w'_1, z\} = z - w'_1.$$

Similarly,

(ii) if $w'_1 \geq z \geq w_1 \geq w_2$, we have $0 \leq w'_1 - z$;

(iii) if $w'_1 \geq w_1 \geq z \geq w_2$, we have $w_1 - z \leq w'_1 - z$, and

(iv) if $w'_1 \geq w_1 \geq w_2 \geq z$, we have $w_1 - w_2 \leq w'_1 - z$.

Since all weak inequalities above hold strictly with strictly positive probabilities, we can conclude that

$$\pi(\sigma_1; \sigma) + \pi(\sigma_2; \sigma) < \pi(\sigma'_1; \sigma').$$

The proof for the case $w_1 < w_2$ follows similar lines, and is therefore omitted. Therefore, no coalition profile such that $\sigma_2 \neq \varnothing$ is stable. To conclude the proof, observe that, as in part (b) of Lemma 1, in the optimized values model the grand coalition always achieves the maximum feasible payoff in this market, i.e., the sum of the $\overline{n}$ highest valuations across all N firms (i.e., $\sum_{i=1}^{\overline{n}} V_{(N,i)}$). Therefore, it has to be stable. ■

Let us now turn to the limited-value case. Allowing joint deviation would in general require conditions stronger than the ones described in part (a) of Lemma 1 for the grand coalition to be stable. That would sometimes yield environments in which the grand coalition is uniquely stable (as in the scenarios illustrated in part (a) of Example 1 in the main paper). *However, it would still be possible to find environments in which other coalition profiles are stable, and sometimes uniquely stable.* To see it, consider the example illustrated in part (b) of Example 1 in the main paper. It is easy to verify that the result is robust to joint deviations.