Online Appendix: Procurement when both price and quality matter

Theory

This document contains contains the complete proof of theorem 1 and a detailed exposition of the recall game.

Appendix Z: Full Proofs

(Not for publication)

Appendix Z: Proof of Theorem 1

Set-up of the Optimal Program

For future reference, this appendix reproduces the optimization problem of the buyer with $U_{hH} = 0$ (Lemma 2) and with the subset of the IC constraints that happen to bind at the optimum.

$$\max_{\{x_k, q_k, U_k\}} \alpha_{lH}\left[x_{lH}W_{lH}(q_{lH}) - U_{lH}\right] + \alpha_{hH}x_{hH}W_{hH}(q_{hH}) + \alpha_{hL}\left[x_{hL}W_{hL}(q_{hL}) - U_{hL}\right] + \alpha_{lL}\left[x_{lL}W_{lL}(q_{lL}) - U_{lL}\right]$$

subject to:

$$
\begin{aligned}
U_{lH} &\geq x_{hH}\Delta\theta_1 && \text{(IC 1)}\\
U_{hL} &\geq U_{lH} - x_{lH}\left[W_{lH}(q_{lH}) - W_{hL}(q_{lH})\right] && \text{(IC 2)}\\
U_{hL} &\geq x_{hH}q_{hH}\Delta\theta_2 && \text{(IC 3)}\\
U_{lL} &\geq U_{lH} + x_{lH}q_{lH}\Delta\theta_2 && \text{(IC 4)}\\
U_{lL} &\geq U_{hL} + x_{hL}\Delta\theta_1 && \text{(IC 5)}\\
U_{lL} &\geq U_{hH} + x_{hH}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2 && \text{(IC 6)}\\
N\sum_{k\in K}\alpha_k x_k &\leq 1 - (1 - \sum_{k\in K}\alpha_k)^N \text{ for all subsets } K \text{ of } \{lH, hH, hL, lL\} && \text{(feasibility)}
\end{aligned}
$$

(We omit the non exclusion constraint). The associated Lagrangian is given by:

$$
\begin{aligned}
&\alpha_{lH}\left[x_{lH}W_{lH}(q_{lH}) - U_{lH}\right] + \alpha_{hH}x_{hH}W_{hH}(q_{hH}) + \alpha_{hL}\left[x_{hL}W_{hL}(q_{hL}) - U_{hL}\right] + \alpha_{lL}\left[x_{lL}W_{lL}(q_{lL}) - U_{lL}\right] && (1)\\
&+\lambda_1\left[U_{lH} - x_{hH}\Delta\theta_1\right] + \lambda_2\left[U_{hL} - U_{lH} + x_{lH}\left(W_{lH}(q_{lH}) - W_{hL}(q_{lH})\right)\right]\\
&+\lambda_3\left[U_{hL} - x_{hH}q_{hH}\Delta\theta_2\right] + \lambda_4\left[U_{lL} - U_{lH} - x_{lH}q_{lH}\Delta\theta_2\right] + \lambda_5\left[U_{lL} - U_{hL} - x_{hL}\Delta\theta_1\right]\\
&\lambda_6\left[U_{lL} - x_{hH}\Delta\theta_1 - x_{hH}q_{hH}\Delta\theta_2\right] - \sum\gamma_K\left[N\sum_{k\in K}\alpha_k x_k - 1 + (1 - \sum_{k\in K}\alpha_k)^N\right]
\end{aligned}
$$

(where λ_i is the Lagrangian multiplier associated with IC constraint i, and γ_K is the multiplier associated with feasibility constraint K). Figure 13 provides a graphical representation of these IC constraints together with their associated multipliers. A dotted line means that a constraint may bind at the optimum. A full line means it always binds.

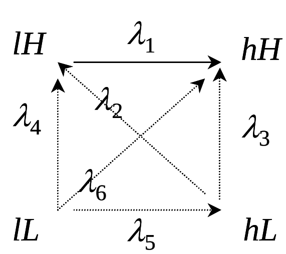

Figure 13: Potentially binding constraints at the solution

The Kuhn-Tucker conditions of this program are standard. For future reference, we only reproduce those with respect to U_k :

$$\lambda_1 - \lambda_2 - \lambda_4 = \alpha_{lH} \tag{2}$$

$$\lambda_2 + \lambda_3 - \lambda_5 = \alpha_{hL} \tag{3}$$

$$\lambda_4 + \lambda_5 + \lambda_6 = \alpha_{lL} \tag{4}$$

Characterization of the Optimal Buying Mechanism

Preliminaries

We first define the notation that we will be using for some of the x_k variables when they take specific values. When x_{lH} takes its maximum value conditional on lL keeping priority in the contract allocation, we will denote it $x_{lH}^{\max}$. Formally, $x_{lH}^{\max}$ is defined by the equation

$$N\left(\alpha_{lH}x_{lH}^{\max} + \alpha_{lL}x_{lL}^{FB}\right) = 1 - (\alpha_{hL} + \alpha_{hH})^N$$

By Border (1991), this implies the following allocation: When there is a type lL, give the contract to lL, if not, give priority to a type lH if there is one. Conversely, $x_{hL}^{\min}$ corresponds to the expected probability of winning for hL when lH and lL have priority over hL (but hL maintains priority over hH). Formally,

$$N\left(\alpha_{lH}x_{lH}^{\max} + \alpha_{hL}x_{hL}^{\min} + \alpha_{lL}x_{lL}^{FB}\right) = 1 - \alpha_{hH}^N$$

Finally, $\overline{x}$ is defined such that $x_{lH} = x_{hL}$ and they have priority over hH in the allocation, that is

$$N\left((\alpha_{lH} + \alpha_{hL})\overline{x} + \alpha_{lL}x_{lL}^{FB}\right) = 1 - \alpha_{hH}^N$$

The proof of Theorem 1 uses the following result repeatedly:

Lemma 7: *Suppose $U_{lH} = x_{hH}\Delta\theta_1$. (1) Suppose further that $U_{hL,lH} \geq U_{hL,hH}$. Then, $x_{hL} > x_{lH}$ if and only if $U_{lL,hL} > U_{lL,lH}$. (2) Suppose now that $U_{hL,lH} \leq U_{hL,hH}$. Then $U_{lL,hL} \geq U_{lL,lH}$ when $x_{hL} \geq x_{lH}$.*

Proof: The result follows directly from a comparison of $U_{lL,lH}$ and $U_{lL,hL}$ (when $U_{hL,lH} \geq U_{hL,hH}$) :

$$U_{lL,lH} = x_{lH}q_{lH}\Delta\theta_2 + x_{hH}\Delta\theta_1 \quad U_{lL,hL} = x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_1 + x_{lH}q_{lH}\Delta\theta_2 + x_{hH}\Delta\theta_1$$

When $U_{hL,hH} \geq U_{hL,lH}$, $U_{lL,hL} = x_{hL}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2$. Since $U_{hL,hH} \geq U_{hL,lH}$ is equivalent to $x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] \leq x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$, the condition implies $U_{lL,hL} \geq U_{lL,lH}$ when $x_{hL} > x_{lH}$. QED.

Lemma 8: *Suppose that $IC_{hL,hH}$ is satisfied. Then $x_{hL} \geq x_{hH} \Longrightarrow IC_{lL,hH}$ is satisfied.*

Proof: IC$_{hL,hH}$ satisfied means that $U_{lL,hL} \overset{\text{defn}}{=} U_{hL} + x_{hL}\Delta\theta_1 \geq U_{hH} + x_{hH}\Delta\theta_2 q_{hH} + x_{hL}\Delta\theta_1$. On this other hand, $U_{lL,hH} = U_{hH} + x_{hH}\Delta\theta_2 q_{hH} + x_{hH}\Delta\theta_1$. Clearly, $U_{lL,hH} \leq U_{lL,hL}$ as long as $x_{hL} \geq x_{hH}$. QED

We are now ready to prove theorem 1. The proof proceeds by progressively partioning the space of parameters into sets of parameters for which the solution shares the same binding IC and feasibility constraints. The logic of the proof is pretty simple, even if the mechanics can be involved. For this reason an exhaustive exposition of the proof of part I, scenario 1 is presented. The arguments in the rest of the proof are presented more briefly where they mirror those in part I, scenario 1.

Proof of part I of Theorem 1: $W_{lH}(\overline{q}) - W_{hL}(\overline{q}) > 0$ i.e. $\Delta\theta_1 > \overline{q}\Delta\theta_2$

The binding constraints in the buyer-optimal efficient mechanism are IC$_{lH,hH}$, IC$_{hL,hH}$ and IC$_{lL,hL}$. The buyer's resulting expected utility is given by

$$\begin{aligned}&\alpha_{lH}x_{lH}W_{lH}(q_{lH}) + \alpha_{hH}x_{hH}[W_{hH}(q_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}\Delta\theta_2] \\ &+\alpha_{hL}x_{hL}[W_{hL}(q_{hL}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] + \alpha_{lL}x_{lL}W_{lL}(q_{lL})\end{aligned} \tag{5}$$

(where, again, we have highlighted the virtual welfares associated with each type). Keeping the probabilities fixed at $x_k = x_k^{FB}$, optimizing the q's in (5) requires that only q_{hH} be adjusted away from the efficient level and set equal to

$$q_{hH}^2 = \arg\max\{W_{hH}(q_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}\Delta\theta_2\} \tag{6}$$

This reduces the informational rents of hL and lL. From Lemma 7(2), we know that $U_{lL,hL} \geq U_{lL,lH}$ as long as $U_{hL,hH} \geq U_{hL,lH}$. Hence, we need to consider only two scenarios:

Scenario 1: At q_{hH}^2, $U_{hL,hH} \geq U_{hL,lH}$, that is,

$$x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] \leq x_{lH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] \tag{7}$$

In this case, all IC constraints remain satisfied as we decrease q_{hH} to q_{hH}^2.
We now consider the optimization of the probabilities of winning. From (5) and the model assumptions, the virtual welfare associated with lL is the largest. Moreover, the virtual welfare associated with lH is larger than that associated with hH. Thus, we need to consider three cases depending on the relative ranking of the virtual welfare of hL with respect to the virtual welfares of hH and lH.

1. $VW_{hL} \geq VW_{lH} \geq VW_{hH} : W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{lH}(\overline{q}) > W_{hH}(q^2_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}} q^2_{hH}\Delta\theta_2$

   [Solution 1.1.a]

   The optimal probabilities of winning are $x_k = x_k^{FB}$ since the ranking of the virtual welfares corresponds to the ranking of the first best welfares. All IC constraints are satisfied given the arguments above. The x's and q's are optimized given the binding constraints; $q_{lH} = \overline{q}$, $q_{hH} = q^2_{hH}$ and $q_{hL} = q_{lL} = \underline{q}$.

2. $VW_{lH} > VW_{hL} \geq VW_{hH} : W_{lH}(\overline{q}) > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{hH}(q^2_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}} q^2_{hH}\Delta\theta_2$

   In this case, type lH generates a higher level of virtual welfare than type hL. Thus, the buyer would rather give the contract to supplier lH than to supplier hL, i.e. he would like to change the order of priority in the allocation. Increasing x_{lH} while decreasing x_{hL} concurrently (keeping $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL}^{FB}$ constant) does not initially affect any of the virtual welfare and it increases the buyer's expected utility. This process continues until either a new IC constraint binds or we have reach $x_{lH} = x_{lH}^{\max}$.

   We now argue that the only potentially new binding constraint is $\text{IC}_{lL,lH}$. To see this consider the following:

   (a) <u>hL's IC constraints:</u> Given that $U_{hL,lH} = U_{lH} - x_{lH}[\Delta\theta_1 - \Delta\theta_2\overline{q}]$ and that U_{lH} is not affected by the process, the incentives for hL to imitate lH have actually decreased. $\text{IC}_{hL,lL}$ remain satisfied as well since $\text{IC}_{lL,hL}$ is binding and $x_{lL} > x_{hL}$.

   (b) <u>lH's IC constraints:</u> Because $U_{lH,hL} = U_{hL} + x_{hL}(\Delta\theta_1 - \Delta\theta_2 q^2_{hH})$ and $U_{lH,lL} = U_{hL} + x_{hL}\Delta\theta_1 - x_{lL}\Delta\theta_2\underline{q}$, the incentives for lH to imitate hL and lL have decreased ($U_{hL} = x_{hH}\Delta\theta_1 q^2_{hH}$ is not affected by the process).

   (c) <u>hH's IC constraints:</u> hH continues to have no incentive to imitate hH, hL or lL given that $\text{IC}_{lH,hH}$ and $\text{IC}_{hL,hH}$ are binding, and $U_{hH,lL}$ is not affected by the process.

   (d) <u>lL's IC constraint:</u> By Lemma 8, $\text{IC}_{lL,hH}$ is not affected by the process. By Lemma 7(2), $\text{IC}_{lL,lH}$ remains satisfied as long as $x_{lH} \leq x_{hL}$, but it could start binding afterwards.

   Thus, we continue to increase x_{lH} at the cost of x_{hL} until either $x_{lH} = x_{lH}^{\max}$ or $\text{IC}_{lL,lH}$ starts binding, whichever comes first.

   (a) $x_{lH} = x_{lH}^{\max}$ first. **[Solution 1.1.b]**

   This means that $U_{lL,hL} \geq U_{lL,lH}$ even when x_{lH} reaches its maximum. This corresponds to the solution because there are no more opportunities to increase the buyer's expected utility: the q's are optimized given the binding IC constraints, the x's are optimized given the virtual welfare and the feasibility constraints. The solution is thus: $q_{lH} = \overline{q}$, $q_{hH} = q^2_{hH}$, $q_{hL} = q_{lL} = \underline{q}$ and $x_{lL} = x^{FB}_{lL} > x_{lH} = x^{\max}_{lH} > x_{hL} = x^{\min}_{hL} > x_{hH} = x^{FB}_{hH}$. By the argument just above, all IC constraints are satisfied.

(b) $\text{IC}_{lL,lH}$ starts binding. **[Solution 1.1.c]**

At that point, $U_{lL,lH} = U_{lL,hL}$, that is, $x^{FB}_{hH}[W_{lH}(q^2_{hH}) - W_{hL}(q^2_{hH})] = x_{hL}\Delta\theta_1 - x_{lH}\overline{q}\Delta\theta_2$ (note that by Lemma 7(2), this happens at $x_{lH} > x_{hL}$).

We now argue that we should be looking for a solution where both $\text{IC}_{lL,hL}$ and $\text{IC}_{lL,lH}$ are binding. Indeed, if only $\text{IC}_{lL,lH}$ binds, the virtual welfare associated with hL is W^{FB}_{hL} which is greater than the virtual welfare associated with lH. Thus the buyer would want to set x_{hL} back to x^{FB}_{hL}, but this would bring us back to the starting point.

Thus the buyer further increases his expected utility by increasing x_{lH} and decreasing x_{hL} while keeping $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x^{FB}_{lL}$ constant *and* $x^{FB}_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{hL}\Delta\theta_1 - x_{lH}\overline{q}\Delta\theta_2$. This requires that we adjust q_{hH} (specifically we need to increase q_{hH}).

A change in q_{hH} corresponds to a change in the value of the Lagrangian multiplier on the $\text{IC}_{lL,lH}$ constraint. Using the expressions in (1) to (4), we can rewrite the expressions for lH and hH's virtual welfares as follows:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{\lambda_4}{\alpha_{lH}}q_{lH}\Delta\theta_2\} \tag{8}$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\alpha_{lH} + \lambda_4}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL} + \alpha_{lL} - \lambda_4}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \tag{9}$$

where λ_4 is the Lagrangian multiplier on the $\text{IC}_{lL,lH}$ constraint.

Thus, practically, we increase x_{lH} and decrease x_{hL} concurrently to keep $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x^{FB}_{lL}$ constant. This implies a new value for q_{hH} and q_{lH} to ensure that $x^{FB}_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{hL}\Delta\theta_1 - x_{lH}q_{lH}\Delta\theta_2$. These correspond to a new value for λ_4 through (9). Specifically, λ_4 increases.

This process increases the virtual welfare associated with hL, $W_{hL}(\overline{q}) - \frac{\alpha_{lL} - \lambda_4}{\alpha_{hL}}\Delta\theta_1$, and decreases the virtual welfare associated with lH and hH (see (8) and (9)).

It continues until we have either reached the upper bound to x_{lH}, $x^{\max}_{lH}$, or the virtual welfares associated with lH and hL become equal:

$$\max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{\lambda^*_4}{\alpha_{lH}}q_{lH}\Delta\theta_2\} = W_{hL}(\underline{q}) - \frac{\alpha_{lL} - \lambda^*_4}{\alpha_{hL}}\Delta\theta_1$$

whichever comes first. Thus $\lambda_4 \in (0, \lambda^*_4) \subset (0, \alpha_{lL})$ as required by (4).

This defines the solution: $x_{lL} = x^{FB}_{lL} > x^{\max}_{lH} \geq x_{lH} > x_{hL} \geq x^{\min}_{hL} > x_{hH} = x^{FB}_{hH}$, $q_{lL} = q_{hL} = \underline{q}$ and q_{lH} and q_{hH} defined by (8) and (9), q_{lH}, $q_{hH} < \overline{q}$. The x's are optimized given the virtual welfares and the feasibility constraints. The q's are optimized given the binding constraints.

All IC constraints remain satisfied. The arguments for this are the same as those we made above, except for $\text{IC}_{hL,lH}$, which follows because $x^{FB}_{hH}[W_{lH}(q_{hH})-W_{hL}(q_{hH})] \overset{U_{lL,hL}=U_{lL,lH}}{=} x_{hL}\Delta\theta_1 - x_{lH}q_{lH}\Delta\theta_2 < x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$ when $x_{lH} > x_{hL}$.

3. $VW_{lH} \geq VW_{hH} > VW_{hL}$: $W_{lH}(\overline{q}) > W_{hH}(q^2_{hH}) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q^2_{hH}\Delta\theta_2 > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1$.

In this case, the ideal ordering of types in the allocation is $lL \succ lH \succ hH \succ hL$. The buyer increases his expected utility by decreasing x_{hL}, first to the benefit of x_{lH} (that is, keeping $\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x^{FB}_{lL}$ constant), and then to the benefit of x_{hH} (that is, keeping $N(\alpha_{lH}x^{\max}_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x^{FB}_{lL} + \alpha_{hH}x_{hH}) = 1$).

This process initially does not affect any of the virtual welfares until a new IC constraint binds. By the same arguments as in point 2 above, we can establish that the first binding constraint is $\text{IC}_{lL,lH}$. When it binds $x_{hH}[\Delta\theta_1 - \Delta\theta_2 q^2_{hH}] = x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_2\overline{q}$. At this point, $x_{lH} > x_{hL} > x_{hH}$ (the first inequality comes from Lemma 7(2)).

Once this happens, any further improvement requires that we keep $U_{lL,hL} = U_{lL,lH}$ (otherwise, if $U_{lL,hL} < U_{lL,lH}$, $\text{IC}_{lL,hL}$ ceases to bind, the virtual welfare associated with hL bounces back to W^{FB}_{hL} and thus we get back to the starting point). We are thus in a similar situation as in point 2 above. Any further change in the x's requires some changes in the q's and thus in the value of the multiplier on the IC constraints. Using the expressions in (1) to (4), the resulting virtual welfares associated with lH, hH and hL are:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{\lambda_4}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} \tag{10}$$

$$VW_{hH} = \max_{hH}\{W_{hH}(q_{hH}) - \frac{(\alpha_{lH}+\lambda_4)}{\alpha_{hH}}\Delta\theta_1 - \frac{(\alpha_{hL}+\alpha_{lL}-\lambda_4)}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \tag{11}$$

$$VW_{hL} = W_{hL}(\underline{q}) - \frac{\alpha_{lL}-\lambda_4}{\alpha_{hL}}\Delta\theta_1 \tag{12}$$

where $\lambda_4 \in (0, \alpha_{lL})$ is such that $U_{lL,hL} = U_{lL,lH}$ i.e. $x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{hL}\Delta\theta_1 - x_{lH}q_{lH}\Delta\theta_2$ for the current value of x_{hL} (x_{lH} and x_{hH} are well-defined once x_{hL} is defined given that lH has priority hH is also clear). Practically, a decrease in x_{hL} is associated with an increase in q_{hH}, a decrease in q_{lH} and an increase in λ_4. This decreases VW_{lH} and VW_{hH} and increases VW_{hL}.

The difference relative to Solution 1.1.c is what ends this process. Here, the process ends when either a new IC constraint binds or the relative ranking of virtual welfare changes.[1] The only new IC constraint that can bind is $\text{IC}_{lL,hH}$. This happens at $x_{hL} = x_{hH}$. Thus we need to distinguish the following cases depending on which event happens first:

(a) We have reached $VW_{lH} \geq VW_{hH} = VW_{hL}$ and $x_{lH} = x_{lH}^{\max}$. Then this is the solution. The buyer is indifferent between hH and hL. The qualities are given by the value of λ_4 that solves for $VW_{hH} = VW_{hL}$ in (11) and (12), $q_{lL} = q_{hL} = \underline{q}$ and $x_{lL} = x_{lL}^{FB}$, $x_{lH} = x_{lH}^{\max} > x_{hL}^{\min} \geq x_{hL} > x_{hH} \geq x_{hH}^{FB}$. **[Solution 1.1.d]**

(b) We have reached $VW_{lH} \geq VW_{hH} = VW_{hL}$ at $x_{lH} < x_{lH}^{\max}$. Then the buyer can further increase his expected utility by decreasing x_{hL} and increasing x_{lH} keeping $U_{lL,lH} = U_{lL,hL}$. This further decreases VW_{lH} and VW_{hH} and increases VW_{hL}. The process stops when either $VW_{lH} = VW_{hL}$ or $x_{lH} = x_{lH}^{\max}$, whichever comes earlier. At the solution the q's are defined from (11) and (12) for the value of λ_4 at which the process stops, $q_{lL} = q_{hL} = \underline{q}$ and $x_{lL} = x_{lL}^{FB}$, $x_{lH}^{\max} \geq x_{lH} > x_{hL} \geq x_{hL}^{\min}$ and $x_{hH} = x_{hH}^{FB}$. This corresponds to **Solution 1.1.c.** above.

(c) We have reached $VW_{lH} = VW_{hH} > VW_{hL}$. (note that this implies that $q_{lH} < q_{hL}$ given (10) and (11)). The buyer further increases his expected utility by decreasing x_{hL} and adjusting x_{lH} and x_{hH} in a way that preserves $VW_{lH} = VW_{hH}$ and $U_{lL,lH} = U_{lL,hL}$.[2] Thus λ_4 is fixed and the virtual welfares are not affected. This process continues until $x_{hL} = x_{hH}$ $(< x_{lH})$ at which point $U_{lL,hH}$ starts binding. At this stage we have:

$$\begin{aligned} U_{lL,lH} &= x_{hH}\Delta\theta_1 + x_{lH}q_{lH}\Delta\theta_2 = U_{lL,hH} = x_{hH}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2 \\ &= U_{lL,hL} = x_{hL}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2 \end{aligned}$$

Using the expressions in (1) to (4), the virtual welfares are given by

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{\lambda_4}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} \tag{13}$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH} - \frac{\alpha_{lH}+\lambda_4}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_4}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \tag{14}$$

$$VW_{hL} = W_{hL}(\underline{q}) - \frac{\alpha_{lL}-\lambda_4-\lambda_6}{\alpha_{hL}}\Delta\theta_1 \tag{15}$$

where λ_4 and λ_6 are the multipliers on the $\text{IC}_{lL,lH}$ and $\text{IC}_{lL,hH}$ constraint respectively.

[1] No feasibility constraint binds in the process. Indeed, the only potential feasibility constraint would involve x_{hH} hitting its maximum but this never occurs before $x_{hH} = x_{hL}$.

[2] The feasibility constraints on the x's are $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hH}x_{hH}) \leq 1 - \alpha_{hL}^N$ and $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$

There exists a value for λ_4 and λ_6 such that $VW_{lH} = VW_{hH} = VW_{hL}$ and $U_{lL,lH} = U_{lL,hL} = U_{lL,hH}$ and $N(\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL}^{FB} + \alpha_{hH}x_{hH}) = 1$. Indeed, we have five equations and five unknowns: λ_4, λ_6, x_{lH}, x_{hL} and x_{hH} (from (15) and the fact $VW_{lH} = VW_{hL}$, we know that $\alpha_{lL} - \lambda_4 - \lambda_6 > 0$, thus $\alpha_{hL} + \alpha_{lL} - \lambda_4$ in (14) is ensured to be positive which is required by the non negative constraint on the multipliers).

These values for λ_4 and λ_6 correspond to the solution. At the solution, $x_{lH} > x_{hH} = x_{hL}$ (implied by $U_{lL,lH} = U_{lL,hL} = U_{lL,hH}$), $q_{lH} < q_{hL} < \overline{q}$ and $q_{hL} = q_{hH} = \underline{q}$. The buyer is indifferent among lH, hH and hL and the x's are thus optimized. The q's are optimized given the binding constraints and the value of the multipliers. No new constraint binds in the process. The argument for this is identical as the one in point 2, except for $\text{IC}_{hL,lH}$, which follows because $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] \overset{U_{lL,hL}=U_{lL,lH}}{=} x_{hL}\Delta\theta_1 - x_{lH}q_{lH}\Delta\theta_2 < x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$ when $x_{lH} > x_{hL}$. **[Solution 1.1.e]**

(d) We have reached $x_{hH} = x_{hL}$. At this point, $\text{IC}_{lL,hH}$ starts binding. The rest of the argument is as in point c above: There exists a value for λ_4 and λ_6 such that $VW_{lH} = VW_{hH} = VW_{hL}$ and $U_{lL,lH} = U_{lL,hL} = U_{lL,hH}$ and $N(\alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{lL}x_{lL}^{FB} + \alpha_{hH}x_{hH}) = 1$. The solution is thus Solution 1.1.e.

Scenario 2: At q_{hH}^2, $U_{hL,hH} < U_{hL,lH}$, that is, $x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] > x_{lH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$.

In this case, $\text{IC}_{hL,lH}$ becomes binding as we decrease q_{hH}. To reduce hL and lL's rents further, one now needs to decrease q_{lH} at the same time as q_{hH} in such a way that $U_{hL,hH} = U_{hL,lH}$, i.e., $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{lH}^{FB}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$. (Note that this implies that $q_{lH} > q_{hH}$.) Formally, using (1) to (4) in Appendix A, we let q_{lH} and q_{hH} solve:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) + \frac{\lambda_2^*}{\alpha_{lH}}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]\} \tag{16}$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\alpha_{lH} + \lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{(\alpha_{hL} + \alpha_{lL} - \lambda_2^*)}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \tag{17}$$

for the value of $\lambda_2^* \in (0, \alpha_{hL} + \alpha_{lL})$ such that $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{lH}^{FB}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$ (λ_2 is the multiplier on $\text{IC}_{hL,lH}$). Such value for λ_2 always exists. When $\lambda_2^* = 0$, $q_{lH} = \overline{q}$ and $q_{hH} = q_{hH}^2$ so that $x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] > x_{lH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$ from the definition of scenario 2. When $\lambda_2^* = \alpha_{hL} + \alpha_{lL}$, $q_{lH} < q_{hH} = \overline{q}$ and $x_{hH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] < x_{lH}^{FB}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)]$).

Relative to the BOEM, only the rents of hL and lL have decreased. The IC constraint of hL is taken care of by construction, and $U_{lL,hL} \geq U_{lL,lH}$ from Lemma 7(1). Hence, all IC constraints remain satisfied.

We now optimize over the x's. Notice that $VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH})+\frac{\lambda_2^*}{\alpha_{lH}}[W_{lH}(q_{lH})-W_{hL}(q_{lH})]\} > W_{lH}(\overline{q}) > VW_{hH}$. Hence, we need to consider three cases depending on the relative ranking of the virtual welfare associated with hL.

1. $VW_{hL} \geq VW_{lH} > VW_{hH}$: $W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq \max_{q_{lH}}\{W_{lH}(q_{lH}) + \frac{\lambda_2^*}{\alpha_{lH}}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]\}$ **[Solution 1.2.a]**

   The optimal probabilities are thus $x_k = x_k^{FB}$. The values of q_{lH} and q_{hH} are defined in (16) and (17) and $\overline{q} > q_{lH} > q_{hH} > q_{hH}^2$, $q_{hL} = q_{lL} = \underline{q}$.

2. $VW_{lH} > VW_{hL} \geq VW_{hH}$: $\max_{q_{lH}}\{W_{lH}(q_{lH}) + \frac{\lambda_2^*}{\alpha_{lH}}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]\} > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{hH}(q_{hH}^2) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}^2\Delta\theta_2$ (note that the condition is on VW_{hH} evaluated at $\lambda_2 = 0$).

   At the current value of λ_2, the buyer prefers to give the contract to lH over hL. As we progressively increase x_{lH} at the expense of x_{hL}, while keeping $x_{hH}^{FB}[W_{lH}(q_{hH})-W_{hL}(q_{hH})] = x_{lH}[W_{lH}(q_{hL}) - W_{hL}(q_{hL})]$, we decrease λ_2 (i.e. increase q_{lH} and decrease q_{hH} - from (16) and (17)). This decreases VW_{lH} and increases VW_{hH}.

   This process continues until the relative ordering of virtual welfares changes or the binding IC constraints change (at least of one these two events happen before we reach the feasibility constraint $x_{lH} = x_{lH}^{\max}$). Specifically, the two IC constraints we need to worry about are $\text{IC}_{hL,lH}$ which stops binding when $\lambda_2 = 0$, and $\text{IC}_{lL,lH}$ which starts binding when $x_{lH} = x_{hL}$. This yields three cases depending on which event happens first:

   (a) $VW_{lH} = VW_{hL}$ first (note that given the assumption of this case, $VW_{hL} \geq VW_{hH}$ always): We have then reached the solution. At the solution, the probabilities of winning are: $x_{lL} = x_{lL}^{FB} > x_{hL}^{FB} > x_{hL} > x_{lH} > x_{lH}^{FB} > x_{hH} = x_{hH}^{FB}$ where x_{lH} and x_{hL} are defined implicitly by $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$ for the values of q_{hH} and q_{lH} that solve (16) and (17) at the current value of λ_2 ($q_{hH} < q_{lH}$). The x's are optimized given the virtual welfares. The q's are optimized given the binding constraints and the value of λ_2. **[Solution 1.2.b]**

   (b) $\lambda_2 = 0$ first. $\text{IC}_{hL,lH}$ ceases to bind and $q_{hH} = q_{hH}^2$ and $q_{lH} = \overline{q}$. As x_{lH} further increases and x_{hL} decreases, the buyer increases his expected utility. None of the virtual welfares are affected in the process, and thus this continues until we either reach $x_{lH} = x_{lH}^{\max}$ or $\text{IC}_{lL,lH}$ starts binding (this happens when $x_{hH}^{FB}[W_{lH}(q_{hH}^2) - W_{hL}(q_{hH}^2)] = x_{hL}\Delta\theta_1 - x_{hL}\Delta\theta_2\overline{q}$.).

In the first case, we are as in **Solution 1.1.b:** $x_{lL} = x_{lL}^{FB} > x_{lH} = x_{lH}^{\max} > x_{hL} = x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$, $q_{hH} = q_{hH}^2$ and $q_{lH} = \overline{q}$. The x's are optimized given that, by assumption, $VW_{hL} \geq VW_{hH}$.

In the second case, we are as in **Solution 1.1c.** Thus, $x_{lL} = x_{lL}^{FB} > x_{lH}^{\max} \geq x_{lH} > x_{hL} \geq x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$, $q_{lL} = q_{hL} = \underline{q}$ and q_{lH} and q_{hH} defined by (8) and (9), q_{lH}, $q_{hH} < \overline{q}$.

(c) $x_{lH} = x_{hL}$ first. At this point, $\text{IC}_{lL,lH}$ starts binding. Based on the expressions from (1), reworked using the equalities (2) to (4), the associated virtual welfares are given by:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{(\alpha_{hL} + \alpha_{lL} - \lambda_3)}{\alpha_{lH}}\Delta\theta_2 q_{lH} + \frac{\alpha_{hL} + \lambda_5 - \lambda_3}{\alpha_{lH}}\Delta\theta_1\} \quad (18)$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\lambda_3}{\alpha_{hH}}\Delta\theta_2 q_{hH} - \frac{\alpha_{lH} + \alpha_{hL} + \alpha_{lL} - \lambda_3}{\alpha_{hH}}\Delta\theta_1\} \quad (19)$$

$$VW_{hL} = W_{hL}(\underline{q}) - \frac{\lambda_5}{\alpha_{hL}}\Delta\theta_1 \quad (20)$$

There exist values for λ_3 and λ_5 such that (1) $\overline{x}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$ and (2) $VW_{lH} = VW_{hL}$. To see this, note that the progressive adjustment of x_{lH} until $x_{lH} = x_{hL}$ implies that there exists a value for λ_3 that satisfies condition (1). Once λ_3 is fixed, there is a value of λ_5 that ensures condition (2). Indeed for any feasible λ_3, when $\lambda_5 = 0$, the virtual welfare of hL is greater. When $\lambda_5 = \alpha_{lL}$ and $\lambda_2 = \alpha_{hL} + \alpha_{lL} - \lambda_3$, this follows from the fact that we have assume that $VW_{lH} > VW_{hL}$ when $\text{IC}_{lL,lH}$ becomes binding.

Note that $\lambda_2 = \alpha_{hL} - \lambda_3 + \lambda_5$. If the implied λ_2 is positive, this is the solution: $x_{lL} = x_{lL}^{FB} > x_{hL}^{FB} > x_{hL} = \overline{x} = x_{lH} > x_{lH}^{FB} > x_{hH} = x_{hH}^{FB}$ and the q's solving (18) through (20) above for the values of λ_3 and λ_5 that satisfy conditions (1) and (2) (in particular, $q_{lH} > q_{hH}$). The x's are optimized given the virtual welfares: the buyer is indifferent between lH and hL and $VW_{lH} > VW_{hH}$ follows from the comparison between (18) and (19) when $q_{lH} > q_{hH}$. The q's are optimized given the binding constraints and the value of the multipliers. [**Solution 1.2.c**]

If the implied λ_2 is strictly negative, then $\text{IC}_{hL,lH}$ ceases to bind at some point. We are then in the same situation as in **Solution 1.1.c**. At the solution, $x_{lL} = x_{lL}^{FB} > x_{lH}^{\max} \geq x_{lH} > x_{hL} \geq x_{hL}^{\min} > x_{hH} = x_{hH}^{FB}$, $q_{lL} = q_{hL} = \underline{q}$ and q_{lH} and q_{hH} defined by (8) and (9), q_{lH}, $q_{hH} < \overline{q}$.

3. $VW_{lH} > VW_{hH} > VW_{hL} : W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 < W_{hH}(q_{hH}^2) - \frac{\alpha_{lH}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}q_{hH}^2\Delta\theta_2$ (note that the condition is on VW_{hH} evaluated at $\lambda_2 = 0$).

In this case, we ideally want to decrease x_{hL}, first to the benefit of x_{lH} (then, possibly to the benefit of x_{hH}). Doing this while keeping $x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] = x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$, requires that we decrease λ_2 (cf. (16) and (17)). This decreases VW_{lH} and increases VW_{hH}, but given the condition on this case, the ordering of virtual welfares is not affected. Thus, this process continues until, either we reach $\lambda_2 = 0$ (and thus IC$_{hL,lH}$ ceases to bind) or $x_{lH} = x_{hL}$ (and thus IC$_{lL,lH}$ starts binding).

(a) We reach $x_{lH} = x_{hL}$ when $\lambda_2 > 0$: This implies that $IC_{lL,lH}$ becomes binding in the process. Optimizing from now on with constraints $IC_{lH,hH}$, $IC_{lL,lH}$, $IC_{lL,hL}$, $IC_{hL,lH}$ and $IC_{hL,hH}$ binding requires that we keep $x_{lH} = x_{hL}$. The virtual welfares are given by (18), (19) and (20). Like in part 1, scenario 2, case 2c, we proceed by first looking for values of λ_3, λ_5 and q's such that (1) $\overline{x}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}^{FB}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$, i.e. $U_{lL,hL} = U_{lL,lH}$ and $U_{hL,hH} = U_{hL,hH}$ and (2) $VW_{lH} = VW_{hL}$.

    If the implied λ_2 is positive, then this is the solution (solution 1.2.c) because condition (1) implies that $q_{lH} > q_{hH}$, which in turn ensures that $VW_{lH} = VW_{hL} > VW_{hH}$. The x's are optimized, and so are the q's.

    If the implied λ_2 is negative, then we are as in part I, scenario 1, case 3: the binding constraints are IC$_{lL,lH}$, IC$_{lL,hL}$, IC$_{lH,hH}$ and IC$_{hL,hH}$. This leads to solutions 1.1.c, 1.1.d or 1.1.e.

(b) We reach $\lambda_2 = 0$ when $x_{lH} \leq x_{hL}$. We can continue to increase x_{lH} at the expense of x_{hL}, and afterwards if necessary increase x_{hH} at the expense of x_{hL} until $IC_{lL,lH}$ starts binding. ($IC_{hL,lH}$ no longer binds because increasing x_{lH} beyond x_{hL} means that $x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})] < x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$). The case then reduces to part 1, scenario 1, case 3, implying one of solutions 1.1.c, 1.1.d or 1.1.e apply.

Proof of part II of Theorem 1: $W_{lH}(\overline{q}) - W_{hL}(\overline{q}) < 0$ **i.e.** $\Delta\theta_1 < \overline{q}\Delta\theta_2$

The binding constraints in the buyer-optimal efficient mechanism are IC$_{lH,hH}$, IC$_{hL,lH}$ and IC$_{lL,hL}$. The buyer's resulting expected utility is given by

$$\alpha_{lH}x_{lH}[W_{lH}(q_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}\Delta\theta_2] + \alpha_{hH}x_{hH}[W_{hH}(q_{hH}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1]$$
$$+\alpha_{hL}x_{hL}[W_{hL}(q_{hL}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] + \alpha_{lL}x_{lL}W_{lL}(q_{lL}) \tag{21}$$

Keeping the probabilities fixed at $x_k = x_k^{FB}$, optimizing the q's requires that q_{lH} be set equal to

$$q_{lH}^2 = \arg\max\{W_{lH}(q_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}\Delta\theta_2\} \tag{22}$$

This reduces the informational rents of hL and lL. By Lemma 7(1), we know that $U_{lL,hL} > U_{lL,lH}$ as long as $U_{hL,lH} \geq U_{hL,hH}$. Hence, we need to consider only two scenarios, depending on whether $IC_{hL,hH}$ binds at q^2_{LH}:

Scenario 1: At q^2_{lH}, $U_{hL,lH} \geq U_{hL,hH}$, i.e., $x^{FB}_{lH}[W_{lH}(q^2_{lH}) - W_{hL}(q^2_{lH})] \leq x^{FB}_{hH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$ In this case, all IC constraints remain satisfied as we decrease q_{lH} to q^2_{lH}. Note that $W_{lH}(q^2_{lH}) - W_{hL}(q^2_{lH}) \equiv \Delta\theta_1 - \Delta\theta_2 q^2_{lH} < 0$. We now consider the optimization of the probabilities of winning. From (21), the virtual welfare associated with lL is the largest. This leaves four cases depending on the relative ranking of lH, hH and hL:

1. $VW_{hL} \geq VW_{lH} \geq VW_{hH} : [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] \geq [W_{lH}(q^2_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}} q^2_{lH}\Delta\theta_2] \geq [W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1]$ **[Solution 2.1.a]**

   The optimal probabilities of winning are $x_k = x^{FB}_k$ since the ranking of the virtual welfares corresponds to the ranking of the first best welfares. All IC constraints are satisfied. The x's and q's are optimized given the binding constraints.

2. $VW_{lH} > VW_{hH} \geq VW_{hL} : [W_{lH}(q^2_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}} q^2_{lH}\Delta\theta_2] > W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1 \geq [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1]$; or

   $VW_{lH} > VW_{hL} \geq VW_{hH} : [W_{lH}(q^2_{lH}) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}} q^2_{lH}\Delta\theta_2] > [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] \geq W_{hH}(q_{hH}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1$

   The buyer would like to increase x_{lH} at the expense of x_{hL}. Doing this does not affect the supplier hL's IC constraint: $U_{hL,lH} \geq U_{hL,hH}$ corresponds to $x_{lH}[W_{lH}(q^2_{lH}) - W_{hL}(q^2_{lH})] \leq x^{FB}_{hH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$ and $W_{lH}(q^2_{lH}) - W_{hL}(q^2_{lH}) < 0$. Moreover, as long as $x_{hL} > x_{lH}$, the change in x_{lH} does not affect lL's IC constraint either (Lemma 7(1)). Thus, changing x_{lH} does not initially affect the virtual welfares.

   When we reach $x_{lH} = x_{hL} = \overline{x}$, IC$_{lL,lH}$ starts binding since $U_{lL,hL} = x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_1 + x_{lH}q^2_{lH}\Delta\theta_2 + x_{hH}\Delta\theta_1$ and $U_{lL,lH} = x_{lH}q^2_{lH}\Delta\theta_2 + x_{hH}\Delta\theta_1$. Define $\lambda^*_5 \in (0, \alpha_{lL})$, the value of λ_5 that equalizes the virtual welfares associated with lH and hL:

   $$W_{lH}(q^2_{lH}) + \frac{(\alpha_{hL} + \lambda^*_5)}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL} + \alpha_{lL}}{\alpha_{lH}}\Delta\theta_2 q^2_{lH} = W_{hL}(\underline{q}) - \frac{\lambda^*_5}{\alpha_{hL}}\Delta\theta_1 \tag{23}$$

   (from (1) to (4)). Such a value for λ_5 exists. When $\lambda_5 = 0$, the virtual welfare associated with hL is larger. When $\lambda_5 = \alpha_{lL}$, the virtual welfare of lH is bigger by assumption. Note that this process does not affect the virtual welfare associated with hH, which remains unchanged.

   (a) **[Solution 2.1.b]** If at λ^*_5, $VW_{lH} = VW_{hL} > VW_{hH}$, then the solution is $q_{lH} = q^2_{lH}$, $q_{hH} = \overline{q}$ and $q_{hL} = q_{lL} = \underline{q}$ and $x_{lL} = x^{FB}_{lL}, x_{hH} = x^{FB}_{hH}$, and $x_{lH} = x_{hL} = \overline{x}$. All IC constraints are satisfied. The q's and the x's are optimized given the binding constraints (in particular, the buyer is indifferent between lH and hL, but strictly prefer these to hH).

(b) If at λ_5^*, $VW_{lH} = VW_{hL} < VW_{hH}$, the buyer prefers hH to lH or hL. He increases his expected utility by raising x_{hH} while keeping $U_{lL,lH} = U_{lL,hL}$, that is, $x_{lH} = x_{hL}$, and $\lambda_5 = \lambda_5^*$. This process does not initially affect any of the virtual welfares until $\text{IC}_{hL,hH}$ starts binding (this happens at $x_{hL} = x_{lH} > x_{hH}$ given that $q_{lH} = q_{lH}^2 < q_{hH} = \overline{q}$ when $U_{hL,hH} \leq U_{hL,lH}$).

From then on, $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding. The expressions for the resulting virtual welfares are given by:

$$VW_{lH} = \max_{q_{lH}}\{W_{lH}(q_{lH}) - \frac{(\alpha_{hL} + \alpha_{lL} - \lambda_3)}{\alpha_{lH}}\Delta\theta_2 q_{lH} + \frac{\alpha_{hL} + \lambda_5 - \lambda_3}{\alpha_{lH}}\Delta\theta_1\} \quad (24)$$

$$VW_{hH} = \max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\lambda_3}{\alpha_{hH}}\Delta\theta_2 q_{hH} - \frac{\alpha_{lH} + \alpha_{hL} + \alpha_{lL} - \lambda_3}{\alpha_{hH}}\Delta\theta_1\} \quad (25)$$

$$VW_{hL} = W_{hL}(\underline{q}) - \frac{\lambda_5}{\alpha_{hL}}\Delta\theta_1 \quad (26)$$

The buyer increases his expected utility by continuing to increase x_{hH} at the cost of x_{hL} and x_{lH}, while satisfying: (1) $U_{lL,lH} = U_{lL,hL}$ (thus $x_{lH} = x_{hL}$), (2) $U_{hL,hH} = U_{hL,lH}$, that is $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$, and (3) $VW_{lH} = VW_{hL}$. This requires an increase in λ_3 and a decrease in λ_5, i.e. a rise in q_{lH} and a decrease in q_{hH} (nonetheless, $q_{lH}^2 < q_{lH} < q_{hH}$ remains as long as $VW_{lH} \leq VW_{hH}$ as is apparent from (24) and (25)).[3]

This process stops when either $VW_{hH} = VW_{lH} = VW_{hL}$ or we hit a non negativity constraint for the multiplier $\lambda_2 = \alpha_{hL} + \lambda_5 - \lambda_3$.

i. **[Solution 2.1.d]** Suppose $VW_{hH} = VW_{lH} = VW_{hL}$ at a point where $\lambda_2 \geq 0$. Then we have reached the solution. The q's are defined from (24) and (25) for the values of λ_3 and λ_5 that equalize the virtual welfares (note that this implies that $q_{lH} < q_{hH}$, so that, in turn, $U_{hL,hH} = U_{hL,lH}$ implies $x_{lH} > x_{hH}$ as required for incentive compatibility). The x's are such that $x_{lL} = x_{lL}^{FB}$, and $x_{lH}^{FB} > x_{lH} = x_{hL} > x_{hH} > x_{hH}^{FB}$ with $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$.[4] All IC constraints are satisfied. The q's are optimized given the binding constraints. The

[3] Formally, we have four equations (the three constraints mentioned in the text, plus the feasibility constraint $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$) and five unknowns: x_{hH}, x_{hL}, x_{lH} and λ_3 and λ_5 (the q's are determined on the basis of the λ's by (24) and (25)). Thus any value for x_{hH} pins down the other variables.

[4] No other feasibility constraint for the probabilities of winning binds, except for the one-type constraint for x_{lL}.

x's are optimized given the resulting virtual welfares (the buyer is indifferent among lH, hL and hH).

ii. **[Solution 2.1.e]** Suppose λ_2 reaches zero at a point where $VW_{hH} > VW_{lH} = VW_{hL}$.

Let λ_5^{**}, the value of λ_5 at this point. We also have $q_{lH}^2 < q_{lH} < q_{hH}$ and $x_{lH} = x_{hL} > x_{hH}$ at this point. The buyer further increases his utility by increasing x_{hH} at the cost of x_{lH} and x_{hL}, while keeping $U_{lL,lH} = U_{lL,hL}$ and $VW_{lH} = VW_{hL}$ (i.e. $\lambda_5 = \lambda_5^{**}$ and the q's are fixed at $q_{lH} < q_{hH}$).[5] This process at first does not affect the virtual welfares (since λ_5 is fixed, we keep having $VW_{hH} > VW_{lH} = VW_{hL}$), until IC$_{lL,hH}$ starts binding.[6] At this stage we have:

$$\begin{aligned} U_{lL,lH} &= x_{hH}\Delta\theta_1 + x_{lH}q_{lH}\Delta\theta_2 = U_{lL,hH} = x_{hH}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2 \\ &= U_{lL,hL} = x_{hL}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2 \end{aligned}$$

thus $x_{hL} = x_{hH} < x_{lH}$. To keep increasing the buyer's welfare while satisfying all three constraints out of lL requires that we keep $x_{hL} = x_{hH}$. Thus we increase both x_{hL} and x_{hH} at the expense of x_{lH} (this will indeed increase the buyer's utility since $VW_{hH} > VW_{lH} = VW_{hL}$), and adjust the q's as needed, that is, we increase q_{lH} and decrease q_{hH}. We do this until $VW_{lH} = VW_{hL} = VW_{hH}$. We have then reached the solution. At the solution, $q_{lH} < q_{hH}$ and $x_{lL} = x_{lL}^{FB}$, and $x_{lH}^{FB} > x_{lH} > x_{hL} = x_{hH} > x_{hH}^{FB}$ with $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$.

3. $VW_{hL} > VW_{hH} > VW_{lH} : [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] > [W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1] > [W_{lH}(q_{lH}^2) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}^2\Delta\theta_2]$; or

   $VW_{hH} > VW_{hL} > VW_{lH} : W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1 > [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1] > [W_{lH}(q_{lH}^2) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}q_{lH}^2\Delta\theta_2]$

   In this case, the buyer would like to increase x_{hH} at the expense of x_{lH}. As we increase x_{hH} and decrease x_{lH}, we reach a point where $x_{lH}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)] = x_{hH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$, that is, IC$_{hL,hH}$ starts binding.

A candidate solution is defined by the value of $\lambda_2 \in (0, \alpha_{hL} + \alpha_{lL})$ that equates VW_{lH} and

[5] The exact way in which x_{lH} and x_{hL} are decreased is determined by $U_{lL,lH} = U_{lL,hL}$, i.e. $x_{lH}q_{lH}\Delta\theta_2 + x_{hH}\Delta\theta_1 = x_{hL}\Delta\theta_1 + x_{hH}q_{hH}\Delta\theta_2$ and the feasibility constraint $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$.

[6] This is the only constraint that can bind in the process. No new constraint can bind out of lH since $U_{lH} = x_{hH}\Delta\theta_1$ increases and alternatives decrease. No new constraint can bind out of hL because $\Delta\theta_1 - \Delta\theta_2 q_{hH} < 0$ given that $q_{hH} > q_{lH} > q_{lH}^2$ and $VW_{hH} \geq VW_{lH}$.

VW_{hH} :

$$\max_{q_{lH}}\{W_{lH}(q_{lH})+\frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1-\frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} = \max_{q_{hH}}\{W_{hH}(q_{hH})-\frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1-\frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} \quad (27)$$

(from (1) to (4)). Such value for λ_2 exists since the virtual welfare of lH is larger than that of hH at $\lambda_2 = 0$, and smaller at $\lambda_2 = \alpha_{hL} + \alpha_{lL}$ by assumption. By inspection of (27), this happens at $\frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}} < \frac{\lambda_2^*}{\alpha_{lH}}$ that is, the resulting q's are such that $q_{lH}^2 < q_{lH} < q_{hH}$. Finally, we require that $U_{hL,lH} = U_{hL,hH}$, that is, $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$ which implies that $x_{lH} > x_{hH}$ as required by incentive compatibility.

This process only affected VW_{hL} and VW_{hH}. If $VW_{hL} > VW_{hH} = VW_{lH}$ at this point, then this is indeed the solution. The other variables are set such that $x_{lL} = x_{lL}^{FB}$, $x_{hL} = x_{hL}^{FB}$, and $q_{hL} = q_{lL} = \overline{q}$. The q's are optimized given the values of the multipliers and the binding constraints. The x's are optimized given the resulting virtual welfares. All IC constraints are satisfied (IC$_{lL,lH}$ satisfied given Lemma 7(1)). [**Solution 2.1.c**]

If $VW_{hL} < VW_{hH} = VW_{lH}$, the buyer can further increase his expected utility by increasing x_{hH} and x_{lH} at the cost of x_{hL}. He does so while keeping $\lambda_2 = \lambda_2^*$ so that $VW_{hH} = VW_{lH}$. The exact way in which x_{hH} and x_{lH} are increased is pinned down by $x_{lH}[W_{lH}(q_{lH})-W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$. This process does not affect the virtual welfare, until $x_{hL} = x_{lH}$ at which point IC$_{lL,lH}$ starts binding. We are now in a situation where IC$_{lH,hH}$, IC$_{lL,lH}$, IC$_{lL,hL}$, IC$_{hL,lH}$ and IC$_{hL,hH}$ are all binding and $VW_{hL} < VW_{hH} = VW_{lH}$. From then on, the virtual welfares are those defined in (24) - (26). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in 2(b) above, and the proof thus proceeds along the same lines: we look for a solution where IC$_{lH,hH}$, IC$_{lL,lH}$, IC$_{lL,hL}$, IC$_{hL,lH}$ and IC$_{hL,hH}$ are binding and $VW_{hL} = VW_{hH} = VW_{lH}$, or IC$_{lH,hH}$, IC$_{lL,lH}$, IC$_{lL,hL}$, IC$_{lL,hH}$ and IC$_{hL,hH}$ are binding and $VW_{hL} = VW_{hH} = VW_{lH}$. [**Solution 2.1.d or 2.1.e**]

4. $VW_{hH} > VW_{lH} > VW_{hL}$: $W_{hH}(\overline{q}) - \frac{\alpha_{lH}+\alpha_{hL}+\alpha_{lL}}{\alpha_{hH}}\Delta\theta_1 > [W_{lH}(q_{lH}^2) + \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}} q_{lH}^2\Delta\theta_2] > [W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1]$

   Given the ordering of virtual welfares, the buyer is first tempted to increase x_{hH} at the expense of x_{hL}.[7] Two things can happen in the process: (1) IC$_{lL,lH}$ starts binding (this happens at $x_{lH}^{FB} = x_{hL}$ because $U_{lL,lH} = x_{lH}\Delta\theta_2 q_{lH}^2 + x_{hH}\Delta\theta_1$ and $U_{lL,hL} = x_{hL}\Delta\theta_1 - x_{lH}\Delta\theta_1 + x_{lH}\Delta\theta_2 q_{lH}^2 + x_{hH}\Delta\theta_1$), (2) IC$_{hL,hH}$ starts binding (this happens at a point where

[7] That is, keeping the equality $N(\alpha_{lL}x_{lL}^{FB} + \alpha_{lH}x_{lH}^{FB} + \alpha_{hL}x_{hL} + \alpha_{hH}x_{hH}) = 1$.

$x_{hH} < x_{lH}^{FB}$ since $x_{hH}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})] = x_{lH}^{FB}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)]$ at that point, and $W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2) < 0$ from the definition of scenario 1). We examine each case in turn.

(a) $IC_{lL,lH}$ binds first ($x_{lH}^{FB} = x_{hL}$)

Let λ_5^*, the value of λ_5 that equalizes VW_{lH} and VW_{hL}. This was defined in (23). We now have $VW_{hH} > VW_{lH} = VW_{hL}$. Thus the buyer can increase his welfare by increasing x_{hH}. The rest of the solution is as described in 2(b) above. [**Solution 2.1.d or Solution 2.1.e**].

(b) $\text{IC}_{hL,hH}$ binds first:

This happens at $x_{hL} > x_{lH}^{FB} > x_{hH}$ (the first inequality comes from the fact that $\text{IC}_{hL,hH}$ binds first; the second inequality comes from the fact that $q_{lH} < q_{hH} = \overline{q}$ at the point where $\text{IC}_{hL,hH}$ starts binding). Increasing further x_{hH} at the expense of x_{hL}, while keeping $x_{lH}^{FB}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$ requires that we decrease q_{hH} and increase q_{lH}. This corresponds to a rise in λ_3, a decrease in VW_{hH} and an increase in VW_{lH}. This process stops when either $VW_{lH} = VW_{hH}$ or $x_{lH} = x_{hL}$ whichever comes first (note at this stage $x_{lH} = x_{hL} > x_{hH}$ and $IC_{lL,lH}$ starts binding). If $VW_{lH} = VW_{hH}$ first, we can continue to increase the buyer's utility by decreasing x_{hL}, this time to the benefit of both lH and hH while keeping $VW_{lH} = VW_{hH}$ and $U_{hL,hH} = U_{hL,lH}$ (note that this implies $q_{lH} < q_{hH}$ and $x_{hL} > x_{hH}$). This process continues until $x_{hL} = x_{lH}$ at which point $IC_{lL,lH}$ starts binding.

Thus, in both events, we reach a point where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding. From then on, the virtual welfares are those defined in (24) - (26). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in 2(b) above, and the proof thus proceeds along the same lines: we look for a solution where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are binding and $VW_{hL} = VW_{hH} = VW_{lH}$, or $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{lL,hH}$ and $\text{IC}_{hL,hH}$ are binding and $VW_{hL} = VW_{hH} = VW_{lH}$. [**Solution 2.1.d or 2.1.e**]

Scenario 2: At q_{lH}^2, $U_{hL,hH} > U_{hL,lH}$ that is, $x_{lH}^{FB}[W_{lH}(q_{lH}^2) - W_{hL}(q_{lH}^2)] > x_{hH}^{FB}[W_{lH}(\overline{q}) - W_{hL}(\overline{q})]$ In this case, $\text{IC}_{hL,hH}$ becomes binding as we decrease q_{lH} towards q_{lH}^2. To decrease the rents of hL and lL, we now need to decrease q_{lH} and q_{hH}, holding $U_{hL,hH} = U_{hL,lH}$. The optimal q's are defined by:

$$
\begin{aligned}
q^*_{lH} &= \arg\max_{q_{lH}}\{W_{lH}(q_{lH}) + \frac{\lambda^*_2}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda^*_2}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} \\
q^*_{hH} &= \arg\max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\alpha_{lH}+\lambda^*_2}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda^*_2}{\alpha_{hH}}\Delta\theta_2 q_{hH}\}
\end{aligned}
$$

where $\lambda^*_2 \in (0, \alpha_{hL} + \alpha_{lL})$ is chosen such that $x^{FB}_{lH}[W_{lH}(q^*_{lH}) - W_{hL}(q^*_{lH})] = x^{FB}_{hH}[W_{lH}(q^*_{hH}) - W_{hL}(q^*_{hH})]$. Note that the sign of $W_{lH}(q^*_{lH}) - W_{hL}(q^*_{lH}) = \Delta\theta_1 - \Delta\theta_2 q^*_{lH}$ is not pinned down *a priori* so that q_{lH} and q_{hH} cannot be ranked. No other new constraint binds in the process (Lemma 7(1)).

We now consider the optimization of the probabilities of winning. We need to consider five cases:

1. $VW_{hL} \geq VW_{lH} \geq VW_{hH}$: $W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{lH}(q^*_{lH}) + \frac{\lambda^*_2}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda^*_2}{\alpha_{lH}}\Delta\theta_2 q^*_{lH} \geq W_{hH}(q^*_{hH}) - \frac{\alpha_{lH}+\lambda^*_2}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda^*_2}{\alpha_{hH}}\Delta\theta_2 q^*_{hH}$.

   The optimal probabilities of winning are $x_k = x^{FB}_k$. This corresponds to **Solution 1.2.a** except that q_{lH} and q_{hH} cannot be ranked *a priori*.

2. $VW_{lH} > VW_{hL} \geq VW_{hH}$: $W_{lH}(q^*_{lH}) + \frac{\lambda^*_2}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda^*_2}{\alpha_{lH}}\Delta\theta_2 q^*_{lH} > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 \geq W_{hH}(q^*_{hH}) - \frac{\alpha_{lH}+\lambda^*_2}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2}{\alpha_{hH}}\Delta\theta_2 q^*_{hH}$

   $VW_{lH} > VW_{hH} > VW_{hL}$: $W_{lH}(q^*_{lH}) + \frac{\lambda^*_2}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda^*_2}{\alpha_{lH}}\Delta\theta_2 q^*_{lH} > W_{hH}(q^*_{hH}) - \frac{\alpha_{lH}+\lambda^*_2}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda^*_2}{\alpha_{hH}}\Delta\theta_2 q^*_{hH} > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1$

   The buyer would like to increase x_{lH} at the expense of x_{hL}. Doing this while keeping $U_{hL,hH} = U_{hL,lH}$ requires that we adjust the q's and thus λ_2. Specifically, if $\Delta\theta_1 - \Delta\theta_2 q^*_{lH} > 0$, we need to decrease λ_2, otherwise, we need to increase it. In both cases, VW_{lH} goes down and VW_{hH} goes up. This process continues until either a new IC constraint binds or the relative ranking of the virtual welfare changes. Since $x_{lH} > x^{FB}_{lH} > x_{hH}$, the only IC constraint to worry about is $\text{IC}_{lL,lH}$. This gives us three cases to consider depending on which event happens first:

   (a) $VW_{lH} = VW_{hL} \geq VW_{hH}$: We have reached the solution: $x_{lL} = x^{FB}_{lL}$, $x_{hH} = x^{FB}_{hH}$ and $x^{FB}_{hL} > x_{hL} > x_{lH} > x^{FB}_{lH}$ with $N(\alpha_{lL}x^{FB}_{lL} + \alpha_{lH}x_{lH} + \alpha_{hL}x_{hL}) = 1 - \alpha^N_{hH}$, $q_{lL} = q_{hL} = \underline{q}$ and q_{lH} and q_{hH} determined by the value of λ_2 that equates $VW_{hH} = VW_{lH}$. This corresponds to **Solution 1.2.b**.

   (b) $VW_{lH} = VW_{hH} > VW_{hL}$: Note that this means that $q_{lH} < q_{hH}$ and $\Delta\theta_1 - \Delta\theta_2 q_{lH} < 0$ since $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x^{FB}_{hH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$. The buyer continues to increase his expected utility by decreasing x_{hL}, this time, to the benefit of both x_{lH} and x_{hH}, doing so while keeping $VW_{lH} = VW_{hH}$ and $U_{hL,lL} = U_{hL,lH}$. Thus λ_2 is fixed and so are q_{lH} and q_{hH}. Therefore $x_{lH} > x_{hH}$. This process continues until $x_{hL} = x_{lH}$ at which point $\text{IC}_{lL,lH}$ starts binding. From then on, the virtual welfares are those defined in (24) - (26). (note that $\lambda_2 = \alpha_{hL} + \lambda_5 - \lambda_3$). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in scenario 1, 2(b) above ($VW_{hH} > VW_{lH} = VW_{hL}$), and the proof thus proceeds along the same lines. [**Solution 2.1.d or 2.1.e**]

(c) $x_{hL} = x_{lH}$, i.e. $\text{IC}_{lL,lH}$ starts binding. From then on, $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding. The virtual welfares are those defined in (24) - (26). (note that $\lambda_2 = \alpha_{hL}+\lambda_5-\lambda_3$). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. If $VW_{lH} = VW_{hL} > VW_{hH}$, we have reached the solution: $x_{lL} = x_{lL}^{FB}$, $x_{hL}^{FB} > x_{lH} = x_{hL} = \overline{x} > x_{lH}^{FB}$, $x_{hH} = x_{hH}^{FB}$, q_{lH}, $q_{hH} < \overline{q}$ and $q_{lL} = q_{hL} = \overline{q}$. All IC constraints are satisfied and the q's and x's are optimal given the resulting virtual welfares. [**Solution 1.2.c**]

If $VW_{lH} = VW_{hL} < VW_{hH}$, we can conclude that $q_{lH} < q_{hH}$ and $\Delta\theta_1 - \Delta\theta_2 q_{lH} < 0$ since $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}^{FB}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})]$. We are thus in the same situation as in scenario 1, 2(b) above. [**Solution 2.1.d or 2.1.e**]

3. $VW_{hL} > VW_{hH} > VW_{lH} : W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 > W_{hH}(q_{hH}^*) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}^* > W_{lH}(q_{lH}^*) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}^*$.

   (Note that this implies $q_{lH}^* < q_{hH}^*$ and $\Delta\theta_1 - \Delta\theta_2 q_{lH}^* < 0$ given that $x_{lH}^{FB}[W_{lH}(q_{lH}^*) - W_{hL}(q_{lH}^*)] = x_{hH}^{FB}[W_{lH}(q_{hH}^*) - W_{hL}(q_{hH}^*)]$). The buyer wants to increase x_{hH} at the expense of x_{lH}. This requires adjusting λ_2 to maintain the equality $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$. Specifically, λ_2 decreases, q_{lH} increases and q_{hH} decreases, until $VW_{hH} = VW_{lH}$. This occurs at $x_{lH} > x_{hH}$. Indeed, at $x_{lH} = x_{hH}$, $q_{lH} = q_{hH}$ thus $\frac{\alpha_{hL}+\alpha_{lL}-\lambda_2}{\alpha_{hH}}\Delta\theta_2 q_{hH} = \frac{\lambda_2}{\alpha_{lH}}\Delta\theta_2 q_{lH}$ implying that $VW_{hH} < VW_{lH}$. The solution is thus $x_{lL} = x_{lL}^{FB}, x_{hL} = x_{hL}^{FB}$ and $x_{lH}^{FB} > x_{lH} > x_{hH} > x_{hH}^{FB}$ and $q_{lH} < q_{lH} < q_{hH} < \overline{q}$. This corresponds to **solution 2.1.c**

4. $VW_{hH} \geq VW_{lH} \geq VW_{hL} : W_{hH}(q_{hH}^*) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q_{hH}^* \geq W_{lH}(q_{lH}^*) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q_{lH}^* \geq W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1$

   Note that this implies that $q_{lH}^* < q_{hH}^*$ and $\Delta\theta_1 - \Delta\theta_2 q_{lH}^* < 0$. Define $\lambda_2^{**} \in (0, \lambda_2^*)$ such that

$$\max_{q_{hH}}\{W_{hH}(q_{hH}) - \frac{\alpha_{lH}+\lambda_2^{**}}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^{**}}{\alpha_{hH}}\Delta\theta_2 q_{hH}\} = \max_{q_{lH}}\{W_{lH}(q_{lH}) + \frac{\lambda_2^{**}}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^{**}}{\alpha_{lH}}\Delta\theta_2 q_{lH}\} \quad (28)$$

This implies $q^*_{lH} < q_{lH} < q_{hH} < q^*_{hH}$ and $VW_{lH} = VW_{hH} > VW_{hL}$.

$>$From there, the buyer can increase his expected utility by increasing x_{hH} and x_{lH} at the cost of x_{hL}. He does so while keeping $\lambda_2 = \lambda_2^{**}$ so that $VW_{hH} = VW_{lH}$. The exact way in which x_{hH} and x_{lH} are increased is pinned down by $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$. This process does not affect the virtual welfare, until $x_{hL} = x_{lH}$ at which point $\text{IC}_{lL,lH}$ starts binding. We are now in a situation where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding and $VW_{hL} < VW_{hH} = VW_{lH}$. From then on, the virtual welfares are those defined in (24) - (26) (note that $\lambda_2 = \alpha_{hL} + \lambda_5 - \lambda_3$). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in 2(b) above. [**Solution 2.1.d or 2.1.e**]

5. $VW_{hH} > VW_{hL} > VW_{lH}$: $W_{hH}(q^*_{hH}) - \frac{\alpha_{lH}+\lambda_2^*}{\alpha_{hH}}\Delta\theta_1 - \frac{\alpha_{hL}+\alpha_{lL}-\lambda_2^*}{\alpha_{hH}}\Delta\theta_2 q^*_{hH} > W_{hL}(\underline{q}) - \frac{\alpha_{lL}}{\alpha_{hL}}\Delta\theta_1 > W_{lH}(q^*_{lH}) + \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_1 - \frac{\lambda_2^*}{\alpha_{lH}}\Delta\theta_2 q^*_{lH}$

   We are again in a situation where $q^*_{lH} < q^*_{hH}$ and $\Delta\theta_1 - \Delta\theta_2 q^*_{hH} < 0$. The buyer would like to increase x_{hH} at the expense of x_{lH}. Doing so while keeping $x_{lH}[W_{lH}(q_{lH}) - W_{hL}(q_{lH})] = x_{hH}[W_{lH}(q_{hH}) - W_{hL}(q_{hH})]$ requires an adjustment in λ_2, leading to VW_{lH} decreasing and VW_{hH} increasing. This process continues until we reach λ_2^{**} which corresponds to $VW_{lH} = VW_{hH}$ (as defined in (28)). Since $\Delta\theta_1 - \Delta\theta_2 q^*_{hH} < 0$, the corresponding qualities and x's are such that $q^*_{lH} < q_{lH} < q_{hH} < q^*_{hH}$ and $x_{hH} < x_{lH}$.

   We now need to distinguish two cases depending whether $VW_{hL} > VW_{lH} = VW_{hH}$ or $VW_{lH} = VW_{hH} > VW_{hL}$.

   (a) $VW_{hL} > VW_{lH} = VW_{hH}$: Then we have reached the solution: $x_{lL} = x^{FB}_{lL}$, $x_{hL} = x^{FB}_{hL}$ and $x^{FB}_{lH} > x_{lH} > x_{hH} > x^{FB}_{hH}$, $q_{lL} = q_{hL} = \underline{q}$ and $q^*_{lH} < q_{lH} < q_{hH} < q^*_{hH}$ as defined by (28). This corresponds to **Solution 2.1.c.**

   (b) $VW_{lH} = VW_{hH} > VW_{hL}$: the buyer further increases his expected utility by increases x_{lH} and x_{hH} at the expense of x_{hL} while keeping $VW_{hH} = VW_{lH}$ (that is keeping λ_2 and the q's fixed) and $U_{hL,lH} = U_{hL,hH}$ (thus $x_{hH} < x_{lH}$). This process does not affect the virtual welfares until $x_{hL} = x_{lH}$ and $\text{IC}_{lL,lH}$ starts binding. We are now in a situation where $\text{IC}_{lH,hH}$, $\text{IC}_{lL,lH}$, $\text{IC}_{lL,hL}$, $\text{IC}_{hL,lH}$ and $\text{IC}_{hL,hH}$ are all binding and $VW_{hL} < VW_{hH} = VW_{lH}$. From then on, the virtual welfares are those defined in (24) - (26) (note that $\lambda_2 = \alpha_{hL} + \lambda_5 - \lambda_3$). Let λ_5^* such that $VW_{lH} = VW_{hL}$. Since there is no change in λ_3, the q's are not affected ($q_{lH} < q_{hH}$) and the x's implicitly defined by $x_{lH} = x_{hL}$ and $U_{hL,hH} = U_{hL,lH}$ are not affected either. Thus we are exactly in the same situation as in 2(b) above, and the proof thus proceeds along the same lines. [**Solution 2.1.d or 2.1.e**]

Technical Appendix: A Model of Bargaining with Recall

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1 Description of game

Figure 1 gives the order of moves for the case of two suppliers.[1] A supplier's type is known only to that supplier, although the probability of each type is common knowledge. In all other respects the game is one of complete information, in particular all players observe actions made by all players in previous stages of play.

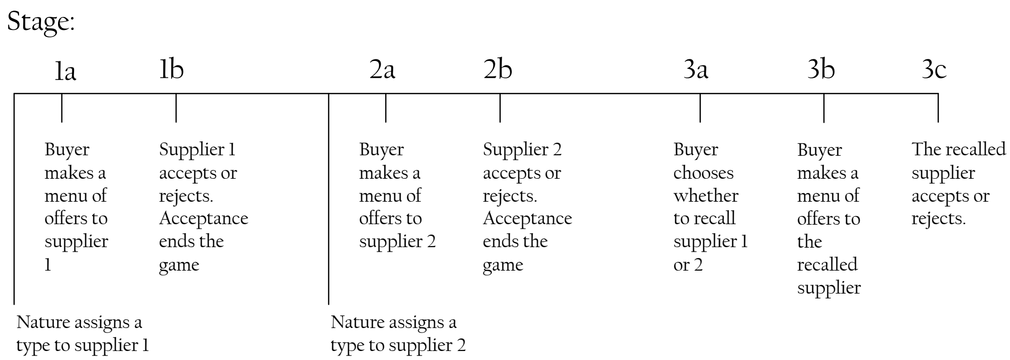

Figure 1: Moves in the game

It is helpful to describe the structure of (multi-node) information sets. At stage 1a, the buyer has an information set since the initial move by nature is not observed. At stage 2a, the buyer arrives at a similar information set where the node that the buyer finds himself at is a function of the types that reject the stage 1 offer and the move by nature assigning types to supplier 2 (since these are independent events we will consider beliefs over them separately). At stage 2b supplier 2 hits

[1] The type of supplier 2 is assigned at the start of stage 2, merely to simplify the writing down of the belief structure of supplier 1 and the buyer.

an information set where the node is determined by the type of supplier 1 conditional on rejecting the stage 1 offer. At stage 3a the buyer hits an information set where the node is determined (conditional on the node at 2a) by the types that reject the stage 2 offer. Lastly at stage 3, supplier 1 arrives at an information set where the node is determined by, again, the types of supplier 2 that reject the stage 2 offer. Clearly, some of these information sets are more interesting from an economic point of view than others.

Ex-ante probabilities of each supplier's types are given by $\alpha_{lL}, \alpha_{lH}, \alpha_{hH}, \alpha_{hL}$. Each supplier knows the order in which they are approached and the history of offers that are made before they are reached, and know that N suppliers exist. Let q_1 describe the quality in the contract intended for the high marginal cost suppliers, let q_2 describe the quality in the contract intended for the low marginal cost suppliers. Let μ^i describe the updated belief of the buyer concerning supplier i's type when supplier i rejects his offer. Hence, μ^i_{lL}, μ^i_{lH}, μ^i_{hL}, μ^i_{hH} denote the updated probabilities that supplier i is of type lL, lH, hL, hH. The buyer buys for sure so that in the stage 3 no exclusion is allowed. That is, we require that the stage 3 offer be acceptable to all types who find themselves at stage 3 with positive probability.

The characterization of the potential equilibria in the game proceed through a series of lemmas. Wherever possible we use the same notation as used in the rest of the paper.

2 Equilibrium concept and approach to solving

We apply the sequential equilibrium concept (see Mas-Colell et al for a definition). In the next sections we explore the form of different types of potential equilibria. The idea is to get to the point where computational derivation of actual equilibrium, given parameter values, is straightforward. We also want to understand the belief structures that support any computed equilibrium and be able to fully articulate strategies.

The approach is to consider stages 3, 2 and 1 in that order. The discussion of stage 3 begins with deriving the offers made and then discusses the choice over whom to recall. We find that in any equilibrium it much be that one of the suppliers is recalled with certainty (ie. there is no randomising over whom to recall). Because of this the discussion of each of stages 2 and 1 is divided into cases in which supplier 1 is recalled with certainty and supplier 2 is recalled with certainty.

After mapping out the form of potential equilibria computation is used to work out, for given parameters, what strategies actually satisfy the equilibrium requirements. The bulk of this work is in working out whom to recall in stage 3 and which types to exclude in stages 2 and 1.

Before we start, it is useful to articulate more specifically what we are looking for in describing potential equilibria: To characterize the properties of a potential equilibria we need to articulate the strategy and beliefs of the buyer and the two suppliers. For the buyer this involves:

(A) Stage 1 offers by the buyer;

(B) Stage 1 and 2 beliefs of the buyer regarding the type of supplier 1 and 2 (respectively);

(C) Stage 2 beliefs of the buyer regarding the type of supplier 1;

(D) Stage 2 offers made by the buyer;

(E) Stage 3 beliefs of the buyer regarding the type of supplier 2;

(F) The decision rule of the buyer as to whom to recall in stage 3; and

(G) Stage 3 offers made by the buyer.

The stage 1 and 2 beliefs of the buyer with respect to the types of supplier 1 and 2 (respectively) are trivial to state: in both stage 1 and 2, the buyer believes that he faces a supplier with one of the four types with probabilities α_{hH}, α_{lH} etc. [2] As a consequence part (B) will not be the focus of the many lemmas that follow.

We also need to articulate:

(H) The decision rule of supplier 1 at stage 1;

(I) The decision rule of supplier 2 at stage 2;

(J) The beliefs of supplier 2 at stage 2 over the type of supplier 1;

(K) The decision rule of each supplier should they be recalled; and

(L) The beliefs of supplier 1 over the type of supplier 2 should supplier 1 be recalled.

Parts (H) and (I) are non-standard only if the supplier is liable to be recalled. The decision rules of suppliers who do not face the possibility of subsequent recall are simply to accept if IC and IR etc are satisfied (ie. the standard rules). As a consequence we will only focus on (H) and (I) when recall is a possibility and will not henceforth consider (K). Parts (J) and (L) are noted as being formally required for a complete characterization of sequential equilibrium. However, they have no economic bearing on play. As a consequence, we merely note that these beliefs should correspond to those of the buyer at the same points.[3]

In the lemmas and discussion that follow we will use the letter notation above to refer to various parts of the equilibrium. This should help keep track of the elements of equilibrium that are being discussed.

[2] Sequential equilibrium is helpful in pinning down these stage 2 beliefs in the case where all types accept the stage 1 offer. No other subtlety exists in this regard.

[3] Setting these beliefs to be the same is not entirely innocuous in off-equilibrium play. As a consequence this is a slight refinement, albeit one used in the literature (see eg. Mas-Collel et al at p452).

3 Stage 3

In this section, we describe the offers made in the third stage and how the buyer decides whom to recall.

3.1 Offers

The buyer's beliefs about the supplier are given by μ_{lL}, μ_{lH},... where μ_{lL} is the probability that the supplier is of type lL. Moreover let $\pi = \mu_{lL} + \mu_{hL}$ (the probability of a low marginal cost supplier). Denote by $(p_i^{\text{recall}}, q_i^{\text{recall}})$, $i = 1, 2$, the third stage offers, where $i = 2$ signals an offer directed at low marginal cost types. The following lemma follows from existing results in the main paper:

Lemma 1: *(G) Stage 3 offers made by the buyer:*

(i) If $\pi = 0$ *and* $\mu_{hH} > 0$*, the stage 3 contract is* $p_1^{\text{recall}} = p_2^{\text{recall}} = \overline{\theta}_1 + \overline{\theta}_2\overline{q}, q_1^{\text{recall}} = q_2^{\text{recall}} = \overline{q}$*. The* hH *type has zero profit and the* lH *type makes a profit of* $\Delta\theta_1$*. If* $\pi = 0$ *and* $\mu_{hH} = 0$*,* $p_1^{\text{recall}} = p_2^{\text{recall}} = \underline{\theta}_1 + \overline{\theta}_2\overline{q}, q_1^{\text{recall}} = q_2^{\text{recall}} = \overline{q}$

(ii) If $\pi = 1$ *and* $\mu_{hL} > 0$*, the stage 3 contract is* $p_1^{\text{recall}} = p_2^{\text{recall}} = \overline{\theta}_1 + \underline{\theta}_2\underline{q}, q_1^{\text{recall}} = q_2^{\text{recall}} = \underline{q}$*. The* hL *type has zero profit and the* lL *type makes a profit of* $\Delta\theta_1$*. If* $\pi = 0$ *and* $\mu_{hL} = 0$*,* $p_1^{\text{recall}} = p_2^{\text{recall}} = \underline{\theta}_1 + \underline{\theta}_2\underline{q}, q_1^{\text{recall}} = q_2^{\text{recall}} = \underline{q}$

(iii) If $0 < \pi < 1$ *and* $\mu_{hH} > 0$*, then the solution is the same as in lemma 6 of the main paper, substituting* μ_i *for* α_i*,* $i \in \{lL, lH, hL, hH\}$*. That is,* $p_1^{\text{recall}}(\pi) = \overline{\theta}_1 + \overline{\theta}_2 q_1^{\text{recall}}(\pi)$*,* $q_1^{\text{recall}}(\pi) = \arg\max\{v(q_1) - \overline{\theta}_2 q_1 - \frac{\pi}{1-\pi}\Delta\theta_2 q_1\}$*,* $p_2^{\text{recall}}(\pi) = \overline{\theta}_1 + \underline{\theta}_2 q_2^{\text{recall}}(\pi) + \Delta\theta_2 q_1^{\text{recall}}(\pi)$*, and* $q_2^{\text{recall}}(\pi) = \underline{q}$*.*

(iv) If $0 < \pi < 1$ *and* $\mu_{hH} = 0$*, then the solution is the same as the* $\{lL, lH, hL\}$ *case in table 5 and theorem 3 of the main paper, substituting* μ_i *for* α_i*,* $i \in \{lL, lH, hL, hH\}$*. That is, if we define* $q_1^* = \arg\max\{v(q_1) - \overline{\theta}_2 q_1 - \frac{\pi}{1-\pi}\Delta\theta_2 q_1\}$ *then if* $\Delta\theta_1 - \Delta\theta_2 q_1^* \le 0$*,* $p_1^{\text{recall}}(\pi) = \underline{\theta}_1 + \overline{\theta}_2 q_1^*$*,* $q_1^{\text{recall}}(\pi) = q_1^*$*,* $p_2^{\text{recall}}(\pi) = \underline{\theta}_1 + \underline{\theta}_2\underline{q} + \Delta\theta_2 q_1^*$ *, and* $q_2^{\text{recall}}(\pi) = \underline{q}$*. Alternatively, if* $\Delta\theta_1 - \Delta\theta_2 q_1^* > 0$*, then define* $q_1^{**} = \min\left\{\frac{\Delta\theta_1}{\Delta\theta_2}, \overline{q}\right\}$ *then.*$p_1^{\text{recall}}(\pi) = \underline{\theta}_1 + \overline{\theta}_2 q_1^{**}$*,* $q_1^{\text{recall}}(\pi) = q_1^{**}$*,* $p_2^{\text{recall}}(\pi) = \overline{\theta}_1 + \underline{\theta}_2\underline{q}$ *, and* $q_2^{\text{recall}}(\pi) = \underline{q}$*.*

3.2 Whom to recall

Given offers in periods 1 and 2, if both are rejected the buyer faces a choice as to whom to recall. If we refine our notation, so that μ_i^k is the belief of the buyer at stage 3 as to the probability of supplier k having type i, we can state the general decision rule of the buyer over whom to recall

Lemma 2: *(F) The decision rule of the buyer as to whom to recall in stage 3: Given beliefs* μ_i^1 *and* μ_i^2*,* $i \in \{lL, hL, lH, hH\}$*, the buyer chooses to recall the buyer with the highest expected* *return given the structure of beliefs and the offers as articulated in lemma 1. Should the buyer be indifferent between who to recall at stage 3, the buyer assigns a probability of recalling supplier j of* σ^j such that $\sigma^j \geq 0$ and $\sum_{j=1,2} \sigma^j = 1$.

When $\mu^1_{hH} > 0$ and $\mu^2_{hH} > 0$, we can exploit more structure than in the previous lemma (when $\mu^k_{hH} = 0$ and $\mu^{-k}_{hH} \geq 0$ the neat ordering below cannot be established). This additional structure is articulated in the following lemma:

Lemma 3: *(F) Given $\mu^1_{hH} > 0$ and $\mu^2_{hH} > 0$, π^1 and π^2 summarize the beliefs of the buyer as to the types of suppliers 1 and 2 respectively. If $\pi^1 \geq \pi^2$ then supplier 1 is recalled with certainty. If $\pi^2 > \pi^1$ then supplier 2 is recalled with certainty.*

Proof: First, note that if $\pi^k = 0$ it must be that only one contract is offered in stage 3 (an offer that extracts all surplus from the hH type and sets quality at the first best for high marginal cost types). If $\pi^{-k} > 0$ then two contracts will be offered. If we denote the expected value to the buyer from recalling supplier k to be $V\left(\pi^k\right)$ then it must be that $V\left(\pi^k = 0\right) < V\left(\pi^{-k} > 0\right)$ since two contracts can always replicate one contract.

Now for $\pi \in (0,1]$, (and dropping the k superscript to ease notation), we note that (using lemma 1 for the contract offered)

$$V(\pi) = (1-\pi)\left(v(q_1^{\text{recall}}(\pi)) - \overline{\theta}_1 - \overline{\theta}_2 q_1^{\text{recall}}(\pi)\right) + \pi\left(v(\underline{q}) - \overline{\theta}_1 - \underline{\theta}_2\underline{q} - \Delta\theta_2 q_1^{\text{recall}}(\pi)\right)$$

Now, if π is decreased to $\widehat{\pi} < \pi$ the contracts implicitly defined by $q_1^{\text{recall}}(\pi)$ and $\underline{q}$ remain incentive compatible and individually rational. Hence we can define

$$\widehat{V}(\pi,\widehat{\pi}) = (1-\widehat{\pi})\left(v(q_1^{\text{recall}}(\pi)) - \overline{\theta}_1 - \overline{\theta}_2 q_1^{\text{recall}}(\pi)\right) + \widehat{\pi}\left(v(\underline{q}) - \overline{\theta}_1 - \underline{\theta}_2\underline{q} - \Delta\theta_2 q_1^{\text{recall}}(\pi)\right)$$

Now, after a little algebra,

$$\frac{\partial \widehat{V}(\pi,\widehat{\pi})}{\partial \widehat{\pi}} = \left[v(\underline{q}) - \overline{\theta}_1 - \underline{\theta}_2\underline{q}\right] - \left[v(q_1^{\text{recall}}(\pi)) - \overline{\theta}_1 - \underline{\theta}_2 q_1^{\text{recall}}(\pi)\right] \geq 0 \qquad \text{(since } \underline{q} \text{ is first best)}$$

Hence for $\widehat{\pi} < \pi$, $V(\pi) \geq \widehat{V}(\pi,\widehat{\pi}) > V(\widehat{\pi})$ where the first inequality is strict for $\pi \in (0,1)$ and the final inequality follows from optimality. Hence, the buyer will always want to recall the supplier with the highest π. When π is equal for both suppliers, lemma 2 establishes that supplier 1 is recalled. QED.

4 Potential Equilibrium Paths

In what follows we work through the exercise of filling in the actions and beliefs that player make and have at stages 1 and 2. We divide the work into three classes which correspond to different types of path through the game tree. The first class are those paths which result in supplier 1 never being recalled and supplier 2 being recalled with certainty. That is, paths where the buyer strictly prefers to recall supplier 2 should the recall stage be reached. The second class are those paths in which supplier 1 is recalled with certainty should the recall stage be reached. The last class are those paths which result in the buyer being indifferent as to which supplier to recall.

4.1 Class 1: The buyer strictly prefers recalling supplier 2

4.1.1 Stage 1

When supplier 1 is not recalled the offers made in stage 1 are as described in theorem 3 of the main paper, where the continuation value is determined by lemma 5 above. Since the recall round does not alter revenue when supplier 2 is recalled with certainty, this implies that the offers are exactly as would be the case for the first of two suppliers approached under the bargaining procedure described in theorem 3.

Lemma 4: *(A) Stage 1 offers by the buyer: The offers made in stage 1 are as described in theorem 3 of the main paper for* $n = 2$.

(C) Stage 2 beliefs of the buyer regarding the type of supplier 1: $\mu_i^1 = \frac{\alpha_i}{1-\Pr(k)}$ *for all types* i *in the set of excluded types (the complement of the set* k*), while* $\mu_j^1 = 0$ *for all types* j *that receive an acceptable offer in stage 1. If the stage 1 offer is acceptable to all types then* $\mu_j^1 \in [0, 1]$ s.t. $\sum_j \mu_j^1 = 1$.

4.1.2 Stage 2

In the main paper the discussion of the sequential mechanism (that is, bargaining with no recall) makes it clear that the offers made in a single shot take-it-or-leave environment are a tight upper bound to what can be achieved in an environment where the buyer can make offers in 2 or more stages. Having supplier 2 recalled with certainty merely adds an extra stage but leaves us squarely within the environment considered in the main paper. As a result the contract offers and payoffs are the same as if the buyer only had one stage to make offers (see lemma 2B in the paper).

Lemma 5: *In any equilibrium in which supplier 2 is recalled with certainty:*

(D) Stage 2 offers made by the buyer: $p_1 = \overline{\theta}_1 + \overline{\theta}_2 q_1$, $p_2 = \overline{\theta}_1 + \underline{\theta}_2 q_2 + \Delta\theta_2 q_1$, $q_1 = \arg\max\{v(q) - \overline{\theta}_2 q - \frac{\alpha_{lL}+\alpha_{hL}}{\alpha_{lH}+\alpha_{hH}}\Delta\theta_2 q\}$ *and* $q_2 = \underline{q}$.

(E) Stage 3 beliefs of the buyer regarding the type of supplier 2; The beliefs depend on the decision rule of supplier 2 at stage 2 (outlined below). Under decision rule (i) $\pi \in [\alpha_{lL} + \alpha_{hL}, 1]$ with $\mu_{hH} \geq 0$. *Under decision rule (ii)* $\pi = \alpha_{lL} + \alpha_{hL}$.

(I) The decision rule of supplier 2 at stage 2: Two decisions rules are possible: (i) The offer in stage 2 is accepted if the endogenous IIR constraint is weakly satisfied, otherwise the offer is rejected. That is, $\lambda_{hH} = \lambda_{hL} = \lambda_{lH} = \lambda_{lL} = 0$; *or (ii) The offer in stage 2 is accepted if the endogenous IIR constraint is strictly satisfied, the offer is rejected if the IIR is not satisfied and is rejected with probabilities* λ_{hH}, λ_{hL}, λ_{lH} *and* λ_{lL} *if the IIR is satisfied with equality, such that, if* γ^k *is the proportion of stage k accepted offers that are* (p_2, q_2), λ_{hH}, λ_{hL}, λ_{lH} *and* λ_{lL} *maintain* $\gamma^2 = \gamma^3$.

4.2 Class 2: The buyer strictly prefers recalling supplier 1

4.2.1 Stage 1

When supplier 1 is recalled a discount factor is introduced in that any benefit that supplier 1 may receive from waiting till the recall stage (stage 3) is mitigated by the chance that the game will terminate with the second suppliers accepting an offer in stage 2. This amounts to an endogenous discount factor that allows some screening to be conducted over time as well as via the menu of contracts offered to supplier 1 in stages 1 and 3.

In what follows we work through a characterization of potential equilibria to the game. The additional notation is δ denoting the intertemporal discount factor for supplier 1 (described above). The offer in stage 2, once the set of excluded types is determined is the same as in the sequential procedure. The set of excluded types in stage two determines the discount factor imposed on supplier 1 and thus this aspect of the offer made to supplier 2 interacts with the rest of the model in a way that is difficult to characterize parsimoniously, but relatively easy to deal with computationally. Some care needs to be taken, as this discount rate can be affected by the stage 3 offers, and hence beliefs, since these affect the stage 2 contract via the continuation value. Thus, δ is a shorthand for $\delta(\pi)$.

Preliminary issues Suppose the supplier expects that the buyer will have beliefs μ_{lL}, μ_{lH},.... in the third stage (as before $\pi = \mu_{lL} + \mu_{hL}$). Thus he expects the buyer to make the offer ($p_1^{\text{recall}}(\pi)$, $q_1^{\text{recall}}(\pi)$), ($p_2^{\text{recall}}(\pi), q_2^{\text{recall}}(\pi)$) in stage 3. As before, the stage 3 offer acts as the supplier's outside option (the endogenous IIR constraint). The difference with cases in which supplier 2 gets recalled is that this outside option now differs depending on the fixed costs. For example, type lH will accept offer (p_1, q_1) in stage 1 if and only if

$$p_1 - \underline{\theta}_1 - \overline{\theta}_2 q_1 \geq \delta(\underbrace{\overline{\theta}_1 + \overline{\theta}_2 q_1^{\text{recall}}(\pi)}_{p_1^{\text{recall}}(\pi)} - \underline{\theta}_1 - \overline{\theta}_2 q_1^{\text{recall}}(\pi))$$

which yields

$$\text{IIR}_{lH} : p_1 \geq \underline{\theta}_1 + \delta\Delta\theta_1 + \overline{\theta}_2 q_1 \tag{1}$$

Similarly, we can derive the other IIR constraints:

$$\text{IIR}_{hH} \quad : \quad p_1 \geq \overline{\theta}_1 + \overline{\theta}_2 q_1 \tag{2}$$

$$\text{IIR}_{lL} \quad : \quad p_2 \geq \underline{\theta}_1 + \delta\Delta\theta_1 + \underline{\theta}_2 q_2 + \delta \underbrace{\Delta\theta_2 q_1^{\text{recall}}(\pi)}_{\text{info rent in stage 3}} \tag{3}$$

$$\text{IIR}_{hL} \quad : \quad p_2 \geq \overline{\theta}_1 + \underline{\theta}_2 q_2 + \delta\Delta\theta_2 q_1^{\text{recall}}(\pi) \tag{4}$$

Notice that $\underline{\theta}_1 + \delta\Delta\theta_1 \leq \overline{\theta}_1$. Thus, if IIR_{hL} is satisfied then so is IIR_{lL}, and similarly, if IIR_{hH} is satisfied then so is IIR_{lH}. This suggests that, when $\delta < 1$, the buyer could possibly exclude the high fixed cost suppliers.

Let λ_{lL} be the probability that a supplier of type lL rejects the buyer's offer in stage 1. Define λ_{lH}, λ_{hL} and λ_{hH} similarly

Finally, there is the usual IC constraint that low marginal cost types do not want to take the contract intended for the high marginal cost guys.

$$\text{IC} : p_2 - \underline{\theta}_2 q_2 \geq p_1 - \underline{\theta}_2 q_1 \tag{5}$$

The exact form of the IC constraint depends on the level of p_1 and p_2 and thus in particular on whether some types are excluded in the first period. The IC constraint places some structure on exclusion in period 1. In particular, if IIR_{hH} is satisfied and the IC constraint is satisfied, then all other IIR constraints are satisfied as long as $q_1 \geq \delta q_1^{\text{recall}}$.

Lastly, given the preceding lemmas 1-4, the remaining elements of equilibria we need to characterize are: (A) Stage 1 offers by the buyer; (C) Stage 2 beliefs of the buyer; and (H) The decision rule of supplier 1 at stage 1. Many cases, varying on who is excluded in the stage 1 offers, are possible and we work through each in turn.

Equilibrium where all supplier types sell in stage 1 We derive the properties that an equilibrium where all supplier types sell in stage 1 must have (at this stage there is no guarantee that such an equilibrium exists). Intuitively, this should look like the stage 3 offers because the IR of the hH type is the same in both periods and the offer in stage 3 will satisfy all stage 1 IR constraints because of the discount. We start by stating the following lemma, and then following with a discussion that establishes the lemma:

Lemma 6: *Any equilibrium where the buyer makes an offer acceptable to all suppliers in stage 1, and supplier 1 gets recalled with certainty in stage 3 is such that:*

(A) Stage 1 offers by the buyer: The contracts offered are $q_1^* = \arg\max\left(v(q) - \overline{\theta}_2 q - \frac{\alpha_{hL}+\alpha_{hL}}{\alpha_{hH}+\alpha_{lH}}\Delta\theta_1 q\right)$, $p_1^* = \overline{\theta}_1 + \overline{\theta}_2 q_1^*$, $q_2^* = \underline{q}$, *and* $p_2^* = \overline{\theta}_1 + \underline{\theta}_2 q_2^* + \Delta\theta_2 q_1^*$.

(C) Stage 2 beliefs of the buyer: At stage 2 the beliefs of the buyer are unrestricted so that $\pi \in [\tau, 1]$ *with* $\mu_{hH} \geq 0$, *where* τ *is defined such that* $q_1^* = \delta q_1^{\text{recall}}(\tau)$ *(stage 3 is off the equilibrium path).*

The strategy of supplier 1 is such that:

(H) The offer in stage 1 is accepted if the endogenous IIR constraint is strictly satisfied. If the IIR constraint is strictly violated the offer is rejected. If the IIR is satisfied with equality, $\lambda_{hH} = 0,.$ $\lambda_{hL}, \lambda_{lH}, \lambda_{lL} \in [0,1]$. *If* $\pi = \tau$ *then* $\lambda_{hL} = 0$..

For optimality, we know that IIR_{hH} must be binding leading to $p_1 = \overline{\theta}_1 + \overline{\theta}_2 q_1$. The other two constraints to worry about are the IC constraint and IIR_{hL} :

$$\begin{aligned} \text{IC} &: \; p_2 \geq \overline{\theta}_1 + \underline{\theta}_2 q_2 + \Delta\theta_2 q_1 \\ \text{IIR}_{hL} &: \; p_2 \geq \overline{\theta}_1 + \underline{\theta}_2 q_2 + \delta\Delta\theta_2 q_1^{\text{recall}}(\pi) \end{aligned}$$

The IC constraint binds if $q_1 > \delta q_1^{\text{recall}}$, and the IIR_{hL} constraint binds otherwise. At most type hL and hH are indifferent between buying in stage 1 or in stage 3 (in other words, λ_{lL} and $\lambda_{lH} = 0$). For given values of λ_{hH} and λ_{hL}, the buyer maximizes

$$\max_{q_1} \left\{ \begin{array}{c} ((1-\lambda_{hH})\alpha_{hH} + \alpha_{lH})\left(v(q_1) - \overline{\theta}_1 + \overline{\theta}_2 q_1\right) \\ + ((1-\lambda_{hL})\alpha_{hL} + \alpha_{lL})\left(v(\underline{q}) - p_2\right) \\ + (1-\delta)\lambda_{hH}\alpha_{hH}V^{S2} + \delta\lambda_{hH}\alpha_{hH}\left(v(q_1^{\text{recall}}) - \overline{\theta}_1 - \overline{\theta}_2 q_1^{\text{recall}}\right) \\ + \delta\lambda_{hL}\alpha_{hL}\left(v(\underline{q}) - \overline{\theta}_1 - \Delta\theta_2 q_1^{\text{recall}} - \underline{\theta}_2\underline{q}\right) \end{array} \right\} \tag{6}$$

subject to IIR_{hL} and IC

where V^{S2} is the profit to the buyer realized if supplier 2 is made an offer and accepts. We first claim that at the optimum the IC constraint binds. Indeed, suppose not. Then, $\delta q_1^{\text{recall}} > q_1$ and the IIR_{hL} must bind. This leaves room to increase q_1 by some epsilon increment since this increase the buyer's return from transacting with the high marginal cost type without changing any constraints. Since the limit of this sequence of deviations involves the IC ultimately binding, we conclude that no equilibrium can exist where $\delta q_1^{\text{recall}} > q_1$ and the IC does not bind.

Thus, the contracts are such that $q_1^* = \arg\max\left(v(q) - \overline{\theta}_2 q - \frac{\alpha_{hL}+\alpha_{hL}}{\lambda_{hH}\alpha_{hH}+\alpha_{lH}}\Delta\theta_2 q\right)$, $q_2^* = \underline{q}$ and prices are given by the IIR_{hH} and IC constraints.

Finally we formally show that there cannot be an equilibrium involving mixing by the hH type (that is, $\lambda_{hH} = 0$). The endogenous IIR constraint of the hH type is such that she is indifferent between accepting and rejecting the stage 1 offer. If $\lambda_{hH} \in (0,1)$ and an equilibrium were to have this feature, then the buyer's payoff is

$$V(\lambda_{hH}) = \max_{q_1} \left\{ \begin{array}{c} ((1-\lambda_{hH})\alpha_{hH} + \alpha_{lH}) \left(v(q_1(\lambda_{hH})) - \overline{\theta}_1 - \overline{\theta}_2 q_1(\lambda_{hH})\right) \\ + (\alpha_{hL} + \alpha_{lL}) \left(v(\underline{q}) - \overline{\theta}_1 - \underline{\theta}_2 q_2 - \Delta\theta_2 q_1(\lambda_{hH})\right) \\ + (1-\delta)\lambda_{hH}\alpha_{hH} V^{S2}(\lambda_{hH}) + \delta\lambda_{hH}\alpha_{hH} \left(v(\overline{q}) - \overline{\theta}_1 - \overline{\theta}_2 \overline{q}\right) \end{array} \right\} \quad (7)$$

where we have used that $q_1^{\text{recall}} = \overline{q}$ since there are no low marginal cost type in stage 3 and the buyer's beliefs would be consistent with this. Recall that what $\lambda_{hH} \in (0,1)$ really means is that the hH supplier is playing a strategy that is accept all stage 1 offers that strictly satisfy the endogenous IIR constraint, reject those that strictly violate it and accept those that satisfy with equality $(1-\lambda_{hH})$ of the time. It is easy to show that $\frac{\partial q_1(\lambda_{hH})}{\partial \lambda_{hH}} < 0$, that is, as the proportion of high marginal cost types decrease, the distortion imposed on q_1 by the IC constraint of the low marginal cost types increases. As a consequence, $\left[v(q_1(\lambda_{hH})) - \overline{\theta}_2 q_1(\lambda_{hH})\right]$ is also decreasing in λ_{hH}. Toward establishing a profitable deviation from this proposed equilibrium with mixing, hold the contracts offered in stage 2 fixed, so that $\frac{\partial V(\lambda_{hH})}{\partial \lambda_{hH}}$ is defined as[4]

$$\frac{dV(\lambda_{hH})}{d\lambda_{hH}} = -\alpha_{hH} \left(v(q_1(\lambda_{hH}) - \overline{\theta}_1 - \overline{\theta}_2 q_1(\lambda_{hH})\right) + (1-\delta)\alpha_{hH} V^{S2} + \delta\alpha_{hH} \left(v(\overline{q}) - \overline{\theta}_1 - \overline{\theta}_2 \overline{q}\right)$$

Given this structure, if $\frac{\partial V(\lambda_{hH})}{\partial \lambda_{hH}} < 0$ then the buyer must always have a profitable deviation in which his strategy stays the same, but for an extra epsilon inducement to the seller to accept the stage 1 offer. If $\frac{\partial V(\lambda_{hH})}{\partial \lambda_{hH}} > 0$ then the buyer is better off keeping his strategy as is, but reducing the transfer to the high marginal cost types by epsilon so as to exclude the hH type entirely. If $\frac{\partial V(\lambda_{hH})}{\partial \lambda_{hH}} = 0$ then inspection of the second order conditions reveal that the buyer must be at a local minimum and one of the two deviations noted previously must be profitable. Hence we can rule out any mixing by the hH type. λ_{lL},λ_{lH} and λ_{hL} are unrestricted since their IIR's will be strictly satisfied.

Lastly, the beliefs of the buyer at stage 2 over the types of supplier 1, should the stage 1 offer be rejected, need to be such that the stage 3 offers do not induce supplier 1 to reject the stage 1 offers. Thus π needs to be such that $q_1 \geq \delta q_1^{\text{recall}}(\pi)$. (If $q_1 = \delta q_1^{\text{recall}}(\pi)$, meaning IIR_{hL} is satisfied with equality, then it can be shown that $\lambda_{hL} = 0$ by adapting the argument in subcase 2 of section (4.2.1) below).

Equilibrium when the stage 1 offer is acceptable to lL, lH and hL We now consider the possibility of an equilibrium where, in stage 1, the buyer makes an offer that is acceptable to types lL, lH and hL.

[4] Note that if the stage 2 contractswere not held fixed then a set of measure zero would exist, where the change in the continuation value induced by changing λ_{hH} changes the contracts in stage 2 and, hence, the discount rate δ.

Lemma 7: *Any equilibrium where the buyer makes an offer acceptable to types lL, lH and hL in stage 1, and supplier 1 gets recalled with certainty in stage 3, is such that:*

(A) Stage 1 offers by the buyer: Let (i) $q_1^* = \arg\max\{v(q_1) - \overline{\theta}_2 q_1 - \frac{\alpha_{hL}+\alpha_{lL}}{\alpha_{lH}}\Delta\theta_2 q_1\}$, *(ii)* $p_1^* = \underline{\theta}_1 + \delta\Delta\theta_1 + \overline{\theta}_2 q_1^*$, *(iii)* $q_2^* = q_2^{**} = \underline{q}$, *(iv)* $p_2^* = \underline{\theta}_1 + \delta\Delta\theta_1 + \underline{\theta}_2 q_2^* + \Delta\theta_2 q_1^*$, *(v)* $p_2^{**} = \overline{\theta}_1 + \underline{\theta}_2 q_2^{**} + \delta\Delta\theta_2 q_1^{\text{recall}}(\pi)$, *(vi)* $p_1^{**} = \underline{\theta}_1 + \delta\Delta\theta_1 + \overline{\theta}_2 q_1^{**}$, *and (vii)* $q_1^{**} = \min\left\{(1-\delta)\frac{\Delta\theta_1}{\Delta\theta_2} + \delta q_1^{\text{recall}}(\pi)\ ,\ \overline{q}\right\}$, *then if* $p_2^{**} < p_2^*$ *the optimal contracts to offer are* $\{(p_1^*, q_1^*), (p_2^*, q_2^*)\}$, *else the optimal contracts to offer are given by* $\{(p_1^{**}, q_1^{**}), (p_2^{**}, q_2^{**})\}$.

(C) Stage 2 beliefs of the buyer: $\pi = 0$ *and* $\mu_{hH} > 0$.

The strategy of supplier 1 is such that:

(H) The offer in stage 1 is accepted if the endogenous IIR constraint is strictly satisfied, otherwise the offer is rejected. If the IIR is satisfied with equality, then if $p_2^{**} < p_2^*$*:* $\lambda_{lH} = 0,$ *.and* $\lambda_{hL}, \lambda_{lL}, \lambda_{hH} \in [0,1]$, *otherwise, if* $p_2^{**} \geq p_2^*$, $\lambda_{hH} = \lambda_{hL} = 0,$ .and $\lambda_{lL}, \lambda_{hH} \in [0,1]$

We can argue that IIR_{lH} binds yielding $p_1 = \underline{\theta}_1 + \delta\Delta\theta_1 + \overline{\theta}_2 q_1$. Clearly in such an equilibrium IIR_{lL} is strictly satisfied and IIR_{hH} is strictly violated. The other constraints to consider are IIR_{hL} and IC:

$$\begin{aligned} \text{IC:} &\quad p_2 \geq \underline{\theta}_1 + \delta\Delta\theta_1 + \underline{\theta}_2 q_2 + \Delta\theta_2 q_1 \\ \text{IIR}_{hL} \ : &\quad p_2 \geq \overline{\theta}_1 + \underline{\theta}_2 q_2 + \delta\Delta\theta_2 q_1^{\text{recall}}(\pi) \end{aligned}$$

Unlike in the previous case, we cannot argue a priori that the IC constraint binds (indeed, even if it does not so that $q_1 = \overline{q}$ it can still be the case that $\underline{\theta}_1 + \delta\Delta\theta_1 + \underline{\theta}_2 q_2 + \Delta\theta_2 q_1 < \overline{\theta}_1 + \underline{\theta}_2 q_2 + \delta\Delta\theta_2 q_1^{\text{recall}}(\pi)$). Thus we need to consider two subcases separately.

1. Only IC binds

   Using the fact that the hL type accepts with certainty iif IIR_{hL} does not bind and substituting in the IC constraint, the objective function of the buyer yields:

$$V(\lambda_{lH}) = \max_{q_1} \left\{ \begin{array}{c} (1-\lambda_{lH})\alpha_{lH}\left(v(q_1(\lambda_{lH})) - \underline{\theta}_1 - \delta\Delta\theta_1 - \overline{\theta}_2 q_1(\lambda_{lH})\right) \\ +(\alpha_{lL}+\alpha_{hL})\left(v(\underline{q}) - \underline{\theta}_1 - \delta\Delta\theta_1 - \underline{\theta}_2\underline{q} - \Delta\theta_2 q_1(\lambda_{lH})\right) \\ +\delta(\lambda_{lH}\alpha_{lH}+\alpha_{hH})\left(v(\overline{q}) - \overline{\theta}_1 - \overline{\theta}_2\overline{q}\right) + (1-\delta)(\lambda_{lH}\alpha_{lH}+\alpha_{hH})V^{S2} \end{array} \right\} \quad (8)$$

   Mixing by the lH type can be ruled out by applying arguments made in section (4.2.1), thus $\lambda_{lH} = 0$. Hence, the buyer's expected utility is given by

$$\begin{aligned} V^{\text{3-types}} = &\max_{q_1}\left\{\alpha_{lH}\left(v(q_1) - \underline{\theta}_1 - \delta\Delta\theta_1 - \overline{\theta}_2 q_1\right) + (\alpha_{lL}+\alpha_{hL})\left(v(\underline{q}) - \underline{\theta}_1 - \delta\Delta\theta_1 - \underline{\theta}_2\underline{q} - \Delta\theta_2 q_1\right)\right. \quad (9) \\ &\left.+\delta\alpha_{hH}\left(v(\overline{q}) - \overline{\theta}_1 - \overline{\theta}_2\overline{q}\right) + (1-\delta)\alpha_{hH}V^{S2}\right. \end{aligned}$$

This expression crystallizes the trade-off that the buyer faces. The benefit from excluding the hH type in stage 1 is that it lowers the fixed component of price $(\underline{\theta}_1 + \delta\Delta\theta_1)$ in stage 1. The 'cost' is the chance that the opportunity to buy from type hH will never be reached.

Let the contractual terms be given by $q_2^* = \underline{q}$ and $q_1^* = \arg\max\{v(q_1) - \overline{\theta}_2 q_1 - \frac{\alpha_{lL}+\alpha_{hL}}{\alpha_{lH}}\Delta\theta_2 q_1\}$ with prices, p_1^* and p_2^*, determined by the IIR and IC constraints.

2. IIR_{hL} binds

Letting $q_1^{\text{recall}}(\lambda_{lH}, \lambda_{hL})$ define the quality offered to the high marginal cost type in stage 3, for this case to hold it must be that:

$$p_2^{**} = \overline{\theta}_1 + \underline{\theta}_2 q_2^* + \delta\Delta\theta_2 q_1^{\text{recall}}(0,0) > p_2^*$$

that is, under the contract described in case 1, the IIR constraint of the hL type is violated. This problem can be alleviated by setting the $p_2 = p_2^{**}$. Since $q_2^* = \underline{q}$, and this is invariant to the level of q_1 chosen by the buyer (that is under any combination of these constraints binding $q_2 = \underline{q}$), this gives room to adjust the contract offered to the high marginal cost types in the first period toward the first best. This process stops, either when the IC and IR constraints of the IL type both bind, or when the price offered to the high marginal cost type is at the first best (both contracts impose first best quality and no IC constraints bind).

That is, maximization proceeds by imposing the IIR_{hL} constraint on the usual objective function and maximizing subject to IC also, noting that $q_1 > \overline{q}$ is always dominated by $q_1 = \overline{q}$. This implies $q_1 = \min\left\{(1-\delta)\frac{\Delta\theta_1}{\Delta\theta_2} + \delta q_1^{\text{recall}}(\lambda_{lH}, \lambda_{hL})\ ,\ \overline{q}\right\}$. Note that it need not be the case that the IC constraint between the low and high marginal cost types binds.

Lastly, we need to consider mixing by the lH and hL types. The objective function of the buyer is

$$V(\lambda_{lH}, \lambda_{hL}) = \max_{q_1} \left\{ \begin{array}{c} (1-\lambda_{lH})\alpha_{lH}\left(v(q_1(\lambda_{lH}, \lambda_{hL})) - \underline{\theta}_1 - \delta\Delta\theta_1 - \overline{\theta}_2 q_1(\lambda_{lH}, \lambda_{hL})\right) \\ + ((1-\lambda_{hL})\alpha_{hL} + \alpha_{lL})\left(v(\underline{q}) - \overline{\theta}_1 - \underline{\theta}_2\underline{q} - \delta\Delta\theta_2 q_1^{\text{recall}}(\lambda_{lH}, \lambda_{hL})\right) \\ + (1-\delta)\left(\lambda_{hL}\alpha_{hL} + \lambda_{lH}\alpha_{lH} + \alpha_{hH}\right) V^{S2}(\lambda_{lH}, \lambda_{hL}) \\ +\delta(\lambda_{lH}\alpha_{lH} + \alpha_{hH})\left(v(q_1^{\text{recall}}(\lambda_{lH}, \lambda_{hL})) - \overline{\theta}_1 - \overline{\theta}_2 q_1^{\text{recall}}(\lambda_{lH}, \lambda_{hL})\right) \\ +\delta\left(\lambda_{hL}\alpha_{hL}\right)\left(v(\underline{q}) - \overline{\theta}_1 - \underline{\theta}_2\underline{q} - \Delta\theta_2 q_1^{\text{recall}}(\lambda_{lH}, \lambda_{hL})\right) \end{array} \right\}$$

Dealing with $\lambda_{lH} \in (0,1)$ first, toward establishing a profitable deviation for the buyer, hold $q_1^{\text{recall}}(\lambda_{lH}, \lambda_{hL})$ and $V^{S2}(\lambda_{lH}, \lambda_{hL})$ fixed (so contracts in stages 2 and 3 remain constant as λ_{hL} varies). Taking derivatives, holding stage 2 and 3 contracts fixed, (and after some algebra) yields:[5]

$$\frac{dV(\lambda_{lH},\lambda_{hL})}{d\lambda_{lH}} = -\alpha_{lH}\left(v(q_1) - \underline{\theta}_1 - \overline{\theta}_2 q_1\right) + (1-\delta)\,\alpha_{lH} V^{S2} + \alpha_{lH}\delta\left(v(q_1^{\text{recall}}) - \underline{\theta}_1 - \overline{\theta}_2 q_1^{\text{recall}}\right)$$

If $\frac{dV(\lambda_{lH},\lambda_{hL})}{d\lambda_{lH}} > 0$, then the buyer is better off excluding the lH type. If $\frac{dV(\lambda_{lH},\lambda_{hL})}{d\lambda_{lH}} < 0$ then the buyer is better off by maintaining the same strategy but allocating some extra epsilon transfer if the supplier accepts. If $\frac{dV(\lambda_{lH},\lambda_{hL})}{d\lambda_{lH}} = 0$ then the supplier is better off allocating an extra epsilon transfer if the supplier accepts in stage 1, inducing lH to accept with certainty, and re-optimizing $q_1^{\text{recall}}(\lambda_{lH},\lambda_{hL})$. Thus, it must be that $\lambda_{lH} = 0$.

Next we deal with $\lambda_{hL} \in (0,1)$. As before, toward establishing a profitable deviation for the buyer, hold $q_1^{\text{recall}}(\lambda_{lH},\lambda_{hL})$ and $V^{S2}(\lambda_{lH},\lambda_{hL})$ fixed (so contracts in stages 2 and 3 remain constant as λ_{hL} varies). Taking derivatives, holding stage 2 and 3 contracts fixed, (and after some algebra) yields:

$$\frac{dV(\lambda_{lH},\lambda_{hL})}{d\lambda_{hL}} = (1-\delta)\,\alpha_{hL}\left(V^{S2} - \left(v(\underline{q}) - \overline{\theta}_1 - \underline{\theta}_2\underline{q}\right)\right)$$

If $\frac{dV(\lambda_{lH},\lambda_{hL})}{d\lambda_{hL}} > 0$ (which can happen depending on the form of the contract in stage 2), then the buyer is better off excluding the hL type. If $\frac{dV(\lambda_{lH},\lambda_{hL})}{d\lambda_{hL}} < 0$ then the buyer is better off by maintaining the same strategy but allocating some extra epsilon transfer if the supplier accepts. If $\frac{dV(\lambda_{lH},\lambda_{hL})}{d\lambda_{hL}} = 0$ then the supplier is better off allocating an extra epsilon transfer if the supplier accepts in stage 1, inducing hL to accept with certainty, and re-optimizing $q_1^{\text{recall}}(\lambda_{lH},\lambda_{hL})$ to be first best. Thus $\lambda_{hL} = 0$.

Since this case assumed that if any IC's bind it would be from lL and hL to lH it is necessary to check that lH's IC constraint is satisfied. The problematic case is when $q_1 = \overline{q}$. However the necessary condition for this is $\Delta\theta_1 - \Delta\theta_2\underline{q} \leq 0$ which is always satisfied.

This establishes the initial lemma.

Equilibrium where in stage 1 only the lH- and lL-type suppliers accept **Lemma 8:** *Any equilibrium where the buyer makes an offer acceptable to types lL and lH in stage 1, and supplier 1 gets recalled with certainty in stage 3, is such that:*

[5] Note that the transfer to the low marginal cost suppliers in stage 1, while containing the $q_1^{\text{recall}}(\lambda_{lH},\lambda_{hL})$ term, is actually coming from the strategy of the supplier and thus, for the exercise of looking for deviations by the buyer holding the strategy of the suppliers as fixed, should be treated as exogenously set. Similarly, $q_1(\lambda_{lH},\lambda_{hL})$ is fixed at $q_1(\lambda_{lH},\lambda_{hL}) = \min\left\{(1-\delta)\frac{\Delta\theta_1}{\Delta\theta_2} + \delta\Delta\theta_2 q_1^{\text{recall}}(\lambda_{lH},\lambda_{hL})\ ,\ \overline{q}\right\}$ where the $q_1^{\text{recall}}(\lambda_{lH},\lambda_{hL})$ should be treated as exogenous.

(A) Stage 1 offers by the buyer: Let (i) $q_1^* = \arg\max\{v(q_1) - \overline{\theta}_2 q_1 - \frac{\alpha_{lL}}{\alpha_{lH}}\Delta\theta_2 q_1\}$, *(ii)* $p_1^* = \underline{\theta}_1 + \delta\Delta\theta_1 + \overline{\theta}_2 q_1^*$, *(iii)* $q_2^* = q_2^{**} = \underline{q}$, *(iv)* $p_2^* = \underline{\theta}_1 + \delta\Delta\theta_1 + \underline{\theta}_2 q_2^* + \Delta\theta_2 q_1^*$, *(v)* $p_2^{**} = \underline{\theta}_1 + \delta\Delta\theta_1 + \underline{\theta}_2 q_2^{**} + \delta\Delta\theta_2 q_1^{\text{recall}}(\pi)$, *(vi)* $p_1^{**} = \underline{\theta}_1 + \delta\Delta\theta_1 + \overline{\theta}_2 q_1^{**}$, *and (vii)* $q_1^{**} = \delta\Delta\theta_2 q_1^{\text{recall}}(\pi)$, *then if* $p_2^{**} < p_2^*$ *the optimal contracts to offer are* $\{(p_1^*, q_1^*), (p_2^*, q_2^*)\}$, *else the optimal contracts to offer are given by* $\{(p_1^{**}, q_1^{**}), (p_2^{**}, q_2^{**})\}$.

(C) Stage 2 beliefs of the buyer: $\pi = \frac{\alpha_{hL}}{\alpha_{hL}+\alpha_{hH}}$ *and* $\mu_{hH} > 0$.

The strategy of supplier 1 is such that:

(H) The offer in stage 1 is accepted if the endogenous IIR constraint is strictly satisfied, otherwise the offer is rejected. If the IIR is satisfied with equality, then if $p_2^{**} < p_2^*$*:* $\lambda_{lH} = 0, .$*and* $\lambda_{hH}, \lambda_{hL}, \lambda_{lL} \in [0,1]$, *otherwise, if* $p_2^{**} \geq p_2^*$, $\lambda_{hH} = \lambda_{lL} = 0$ *.and* $\lambda_{hH}, \lambda_{hL} \in [0,1]$.

In such an equilibrium IIR_{lH} must bind yielding $p_1 = \underline{\theta}_1 + \delta\Delta\theta_1 + \overline{\theta}_2 q_1$. The remaining constraints to consider are the IC and IIR_{lL} :

$$\begin{aligned} \text{IC:} &\quad p_2 \geq \underline{\theta}_1 + \delta\Delta\theta_1 + \underline{\theta}_2 q_2 + \Delta\theta_2 q_1 \\ \text{IIR}_{lL} \ : &\quad p_2 \geq \underline{\theta}_1 + \delta\Delta\theta_1 + \underline{\theta}_2 q_2 + \delta\Delta\theta_2 q_1^{\text{recall}}(\pi) \end{aligned}$$

Which one is binding depends on the relationship between q_1 and $\delta q_1^{\text{recall}}$. As before we can argue that the IC constraint must be binding because otherwise a profitable deviation would exist for the buyer. However, we cannot assume that a binding IC implies a slack IIR constraint (normally this would be true since the outside option would be zero for all types, however here the outside option is type specific). Hence we also have to account for the possibility that the IR_{lL} and the IC constraints bind.

1. Only IC binds

   Using the fact that lL accepts with certainty if IIR_{lL} does not bind and substituting in the IC constraint, the objective function of the buyer yields:

$$V(\lambda_{lH}) = \max_{q_1} \left\{ \begin{array}{c} (1-\lambda_{lH})\alpha_{lH}\left(v(q_1(\lambda_{lH})) - \underline{\theta}_1 - \delta\Delta\theta_1 - \overline{\theta}_2 q_1(\lambda_{lH})\right) \\ +\alpha_{lL}\left(v(q_2) - \underline{\theta}_1 - \delta\Delta\theta_1 - \underline{\theta}_2 q_2 - \Delta\theta_2 q_1(\lambda_{lH})\right) \\ +\delta(\lambda_{lH}\alpha_{lH} + \alpha_{hH})\left(v(q_1^{\text{recall}}(\lambda_{lH})) - \overline{\theta}_1 - \overline{\theta}_2 q_1^{\text{recall}}(\lambda_{lH})\right) \\ +(1-\delta)(\lambda_{lH}\alpha_{lH} + \alpha_{hH} + \alpha_{hL}) V^{S2}(\lambda_{lH}) \\ +\delta\alpha_{hL}\left(v(\underline{q}) - \overline{\theta}_1 - \underline{\theta}_2\underline{q} - \Delta\theta_1 q_1^{\text{recall}}(\lambda_{lH})\right) \end{array} \right\} \qquad (10)$$

   Mixing by the lH type can be ruled out by applying arguments made in section (4.2.1), thus $\lambda_{lH} = 0$. Let the contractual terms be given by $q_2^* = \underline{q}$ and $q_1^* = \arg\max\{v(q_1) - \overline{\theta}_2 q_1 - \frac{\alpha_{lL}}{\alpha_{lH}}\Delta\theta_2 q_1\}$ with prices, p_1^* and p_2^*, determined by the IIR and IC constraints.

Finally, we need to make sure that IIR_{hL} is violated. This will be the case if

$$\begin{aligned} p_2 - \overline{\theta}_1 - \underline{\theta}_2 \underline{q} &< \delta(p_2^{\text{recall}} - \overline{\theta}_1 - \underline{\theta}_2 \underline{q}), \text{ that is} \\ \underline{\theta}_1 + \delta\Delta\theta_1 - \overline{\theta}_1 + \Delta\theta_2 q_1 &< \delta\Delta\theta_2 q_1^{\text{recall}}(\pi) \\ \delta(\Delta\theta_1 - \Delta\theta_2 q_1^{\text{recall}}(\pi)) &< \Delta\theta_1 - \Delta\theta_2 q_1 \end{aligned}$$

2. IIR_{lL} and IC bind

Letting $q_1^{\text{recall}}(\lambda_{lH}, \lambda_{lL})$ define the quality offered to the high marginal cost type in stage 3, for this case to hold it must be that

$$p_2^{**} = \underline{\theta}_1 + \delta\Delta\theta_1 + \underline{\theta}_2 q_2^* + \delta\Delta\theta_2 q_1^{\text{recall}}(0,0) > p_2^*$$

reflecting the same intuition as in subcase 2 of section (4.2.1).

That is, maximization proceeds by imposing the IIR_{lL} constraint on the usual objective function and also maximizing subject to IC. This implies $q_1 = \delta q_1^{\text{recall}}(\lambda_{lH}, \lambda_{hL}) < \overline{q}$, $q_2 = \underline{q}$ and prices follow from IIR constraints. Note that it must be the case that the IC constraint between the low and high marginal cost types binds.

Lastly, we need to consider mixing by the lH and hL types. That $\lambda_{lH} = 0$ can be established using arguments that mirror those used in subcase 2 of section (4.2.1). The objective function of the buyer is

$$V(\lambda_{lH}, \lambda_{lL}) = \max_{q_1} \left\{ \begin{array}{c} (1-\lambda_{lH})\alpha_{lH}\left(v(q_1(\lambda_{lH}, \lambda_{lL})) - \underline{\theta}_1 - \delta\Delta\theta_1 - \overline{\theta}_2 q_1(\lambda_{lH}, \lambda_{lL})\right) \\ +(1-\lambda_{lL})\alpha_{lL}\left(v(\underline{q}) - \underline{\theta}_1 - \delta\Delta\theta_1 - \underline{\theta}_2\underline{q} - \delta\Delta\theta_2 q_1^{\text{recall}}(\lambda_{lH}, \lambda_{lL})\right) \\ +(1-\delta)\left(\lambda_{lL}\alpha_{lL} + \lambda_{lH}\alpha_{lH}\right)V^{S2}(\lambda_{lH}, \lambda_{lL}) \\ +\delta(\lambda_{lH}\alpha_{lH} + \alpha_{hH})\left(v(q_1^{\text{recall}}(\lambda_{lH}, \lambda_{lL})) - \overline{\theta}_1 - \overline{\theta}_2 q_1^{\text{recall}}(\lambda_{lH}, \lambda_{lL})\right) \\ +\delta\left(\lambda_{lL}\alpha_{lL} + \alpha_{hL}\right)\left(v(\underline{q}) - \overline{\theta}_1 - \underline{\theta}_2\underline{q} - \Delta\theta_2 q_1^{\text{recall}}(\lambda_{lH}, \lambda_{lL})\right) \end{array} \right\}$$

Next we deal with $\lambda_{lL} \in (0,1)$. As before, toward establishing a profitable deviation for the buyer, hold $q_1^{\text{recall}}(\lambda_{lH}, \lambda_{lL})$ and $V^{S2}(\lambda_{lH}, \lambda_{lL})$ fixed (so contracts in stages 2 and 3 remain constant as λ_{hL} varies). Taking derivatives, holding stage 2 and 3 contracts fixed, (and after some algebra) yields:

$$\frac{dV(\lambda_{lL})}{d\lambda_{lL}} = (1-\delta)\,\alpha_{lL}[\underbrace{V^{S2}}_{\text{positive}} - \underbrace{\left(v(\underline{q}) - \underline{\theta}_1 - \underline{\theta}_2\underline{q}\right)}_{\text{positive}}] \leq 0$$

Where the inequality follows from the fact that $\left(v(\underline{q}) - \underline{\theta}_1 - \underline{\theta}_2\underline{q}\right)$ defines the maximal return to the buyer under any conditions. As a consequence for any $\lambda_{lL} \in (0,1]$ the buyer will always have a profitable deviation in which his strategy stays the same but for an extra epsilon transfer to the seller in stage 1 if the offer is accepted. If $\frac{dV(\lambda_{lL})}{d\lambda_{lL}} = 0$, then the same deviation but reoptimizing $q_1^{\text{recall}}(\lambda_{lH}, \lambda_{lL})$ establishes a strictly preferred deviation.

Since this case assumed that if any IC's bind it would be from lL and hL to lH it is necessary to check that lH's IC constraint is satisfied. The problematic case is when $q_1 = \overline{q}$. However the necessary condition for this is $\Delta\theta_1 - \Delta\theta_2 \underline{q} \leq 0$ which is always satisfied.

Equilibrium where the hL and lL types buy in stage 1 **Lemma 9:** *Any equilibrium where the buyer makes an offer acceptable to types lL and hL in stage 1, and supplier 1 gets recalled with certainty in stage 3, is such that:*

(A) Stage 1 offers by the buyer: $p_2^* = p_1^* = \overline{\theta}_1 + \underline{\theta}_2 q_2^* + \delta\Delta\theta_2\overline{q}$, *and* $q_2^* = q_1^* = \underline{q}$

(C) Stage 2 beliefs of the buyer: $\pi = 0$ *and* $\mu_{hH} > 0$.

The strategy of supplier 1 is such that:

(H) The offer in stage 1 is accepted if the endogenous IIR constraint is strictly satisfied, otherwise the offer is rejected. If the IIR is satisfied with equality, $\lambda_{hL} = 0, .$*and* $\lambda_{hH}, \lambda_{lH}, \lambda_{lL} \in [0, 1]$.

Recall that the IIR constraint of the hL type in stage 1 is given by

$$p_2 \geq \overline{\theta}_1 + \underline{\theta}_2 q_2 + \delta\Delta\theta_2 q_1^{\text{recall}}$$

For any q_1^{recall} and λ_{hL}

$$q_2 = \arg\max_{q_2} [(1 - \lambda_{lL})\,\alpha_{hL} + \alpha_{lL}] \left[v(q_2) - \left(\overline{\theta}_1 + \underline{\theta}_2 q_2 + \delta\Delta\theta_2 q_1^{\text{recall}}(\lambda_{hL})\right)\right]$$

that is, $q_2 = \underline{q}$.

Note that if the hL type is indifferent between stage 1 and stage 3 contracts then it is straightforward to show that the lL type strictly prefers the stage 1 contract. That $\lambda_{lH} = 0$ can be established using arguments that mirror those used in subcase 2 of section (4.2.1)

Hence the stage 1 contracts are: $\left(\overline{\theta}_1 + \underline{\theta}_2\underline{q} + \delta\Delta\theta_2\overline{q}, \underline{q}\right)$. It is easy to verify that the IIR_{lH} is violated by this contract. This implies the initial lemma.

Equilibrium where only the lL type buys in stage 1 **Lemma 10:** *Any equilibrium where the buyer makes an offer acceptable to types lL, lH and hL in stage 1, and supplier 1 gets recalled with certainty in stage 3, is such that:*

(A) Stage 1 offers by the buyer: $p_1^* = p_2^* = \underline{\theta}_1 + \delta\Delta\theta_1 + \underline{\theta}_2 q_2^* + \delta\Delta\theta_2 q_1^{\text{recall}}(\pi)$ *and* $q_1^* = q_2^* = \underline{q}$.

(C) Stage 3 beliefs of the buyer: $\pi = \frac{\alpha_{hL}}{\alpha_{hL}+\alpha_{lH}+\alpha_{hH}}$ *and* $\mu_{hH} > 0$.

The strategy of supplier 1 is such that:

(H) The offer in stage 1 is accepted if the endogenous IIR constraint is strictly satisfied, otherwise the offer is rejected. If the IIR is satisfied with equality, $\lambda_{lL} = 0,$ *.and* $\lambda_{hH}, \lambda_{lH}, \lambda_{hL} \in [0, 1]$ *.*

Recall that the IIR constraint of the lL type in stage 1 is given by

$$p_2 \geq \underline{\theta}_1 + \delta\Delta\theta_1 + \underline{\theta}_2 q_2 + \delta\Delta\theta_2 q_1^{\text{recall}}$$

For any q_1^{recall} and λ_{lL}

$$q_2 = \arg\max_{q_2} \left[(1 - \lambda_{lL})\, \alpha_{lL}\right] \left[v(q_2) - \left(\underline{\theta}_1 + \delta\Delta\theta_1 + \underline{\theta}_2 q_2 + \delta\Delta\theta_2 q_1^{\text{recall}}(\lambda_{lL})\right)\right]$$

that is, $q_2 = \underline{q}$.

That $\lambda_{lL} = 0$ can be established using arguments that mirror those used in subcase 2 of section (4.2.1).

Hence the contracts are $\left(\underline{\theta}_1 + \delta\Delta\theta_1 + \underline{\theta}_2\underline{q} + \delta\Delta\theta_2 q_1^{\text{recall}}(\pi), \underline{q}\right)$ in stage 1.

Equilibrium where no type buys in stage 1 The buyer has the option to make an offer unattractive to all types. In this instance the offer is so high that it is always rejected.

4.2.2 Stage 2

When supplier 2 is not recalled the offers made in stage 2 are as described in theorem 3 of the main paper, but for the fact that the continuation value is determined by lemma 1, above, which determines the form of the contract offered to supplier 1, and the beliefs μ_i^1 which the buyer forms at the start of stage 2. These are beliefs over which type(s) of supplier 1 rejected the stage 1 offer. These beliefs are investigated at length in the discussion of stage 1 below, in lemmas 7 through 11.

Lemma 11: *(D) Stage 2 offers made by the buyer: When supplier 2 is not recalled the offers made in stage 2 are as described in theorem 3 of the main paper for* $n = 2$, *but modifying the continuation value such that it is determined by offers described in lemma 1, above, and the beliefs described in lemmas 6 through 10, above.*

(E) Stage 3 beliefs of the buyer regarding the type of supplier 2; $\mu_i^2 = 1$ *for all types* i *in the set of excluded types, while* $\mu_j^2 = 0$ *for all types* j *that receive an acceptable offer in stage 2. If the stage 2 offer is acceptable to all types then* $\mu_j^2 \in [0, 1]$ *.*

4.3 Class 3: The buyer is indifferent recalling supplier 1 and 2

In this class both supplier 1 and supplier 2 have some probability of being the supplier recalled in the recall stage. This, in effect, creates an endogenous discount rate for both of them in stages 1 and 2 respectively. The proofs and lemmas stated above for Class 2: Stage 1 are easily adapted to these two cases. In the case of the stage 1 actions everything is as stated in lemmas 6 through 10 once it is observed that δ should be replaced by a composite term Γ, where $\Gamma = \sigma^1\delta$. Recall that σ^1 is the probability of recalling supplier 1 in the recall round. In stage 2, lemmas 6 through 10 are similarly adjusted substituting δ for σ^2. In stage 2 the only extra work in the proof is in establishing the no mixing results for the supplier. Since, this requires minor but tedious adjustment of the existing proofs we omit them here.